Chapter 2 Function Spaces

Section 2.1 Complete Vector Spaces

So far, we've worked with finite dimensional subspaces (such as $\mathbb{R}^n$ - for example). And we had a finite set of basis elements $\{x_1, x_2, \ldots, x_n\}$. Now we begin to consider Infinite Dimensional Vector Spaces. Sequence Spaces give us good examples of these

Example 1

Denote the sequence $\{x_n\} = \{x_1, x_2, \ldots\}$ where each $x_i \in \mathbb{R}$ (or $\mathbb{C}$).

$\{x_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ is one example

$\{y_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots\}$ is another example

If we choose sequences with the property that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty, \quad \text{(i.e. form a Convergent Series)}$$

then the function space formed by those sequences is a Vector space! Here is the notation:

$\ell^2 = \left\{ \{x_n\} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ is the vector space referred to as "Little ℓ-two".

Norm: If x denotes the sequence $\{x_n\}$, then the norm of x is defined to be

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$$

So, let $y = \{y_n\} = \{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{3^n}, \ldots\}$, then

$$\|y\| = \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} \left(\frac{1}{3^n}\right)^2\right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{2n}\right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^{n}\right)^{1/2}$$

$$= \left(\frac{1}{1-\frac{1}{9}}\right)^{1/2}$$

Geometric Series from Calc. II

$$= \left(\frac{9}{8}\right)^{1/2} = \frac{3}{2\sqrt{2}}$$

$\|y\| = \frac{3}{2\sqrt{2}} = \frac{3}{4}\sqrt{2}$ So, $y \in \ell^2$

<u>Note:</u> $x = \{x_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ & $\sum_{n=1}^{\infty} |x_n|^2$ is not finite!
So, $x \notin \ell^2$

There is a whole set of ℓ^p-spaces for $p \geq 1$.

$\ell^p = \{\{x_n\} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, and if $x \in \ell^p$

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

The logical choice of a basis for ℓ^p is to use the k^{th} "unit vector". Define $e_k = \{\delta_{ik}\}_{i=1}^{\infty}$; that is,

$$e_k = \{0,0,0,\ldots,0,1,0,\ldots\}$$

↳ k^{th} slot of the sequence contains 1. All other slots are 0's.

Note: $\|e_k\| = 1$

So, the set of sequences that we can use as a basis looks like $\{e_1, e_2, e_3, \ldots, e_k, \ldots\}$

Note that any finite collection of these vectors is Linearly Independent. Why? Choose, $\{e_1, e_2, \ldots, e_N\}$ where N is a fixed integer. Then if

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_N e_N = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \alpha_3 \\ 0 \\ \vdots \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha_N \\ 0 \\ \vdots \end{bmatrix} = 0$$

All of these "vectors" have an infinite # of zeros at the bottom. I only write it like this to point out the similarities between this & working with vectors in $\mathbb{R}^n$.

So, the above eqn implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$

Hence, $\{e_1, e_2, \ldots, e_N\}$ forms a Lin. Ind. set of sequences, and this shows that <u>any</u> <u>finite</u> <u>linear</u> <u>combo</u> of the e_k's forms a Lin. Ind. set. Therefore, the space ℓ^p has an infinite # of Lin. Ind. elements.

Very Important infinite dimensional spaces are Function Spaces – such as $C[0,1]$ or $C^2[a,b]$, etc.

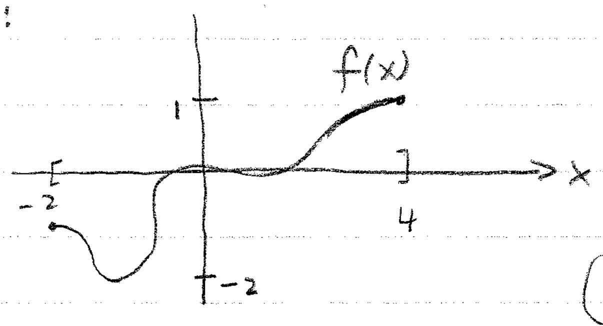
$C[a,b] = \{f(x) : f \text{ is continuous for all } x \in [a,b]\}$

- We can impose several norms (ways to measure distance) on such spaces. If $f \in C[a,b]$, then the
  - Supremum Norm (or 'sup' Norm or "Uniform" Norm) is given by

$$\|f\| = \max_{x \in [a,b]} |f(x)|$$

→ [Sometimes Denoted $\|f\|_\infty$ with subscript]

Picture:



$\|f\| = \max_{x \in [-2,4]} |f(x)|$

so

$\|f\| = 2$

(A measure of how far away from 0 the function's graph gets.)

  - L^2-norm of f is given by

$$\|f\| = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}$$

→ [Sometimes denoted $\|f\|_2$ using subscript]

  - L^p-norm of f is given by

$$\|f\| = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$$

[Sometimes denoted $\|f\|_p$]

- Recall that these definitions of norms satisfy the properties given in Defn. 1.6 (pg. 5 of your textbook).

<u>Claim</u>: $\|f\| = \max_{x\in[a,b]} |f(x)|$ is a norm on $C[a,b]$.

<u>pf</u>: 1. If $f(x) \neq 0$ (that is, if $f(x)$ is not the zero function in $C[a,b]$), then there exists a point $c \in [a,b]$ so that $f(c) \neq 0$. Then $|f(c)| > 0$, and $0 < f(c) \leq \max_{x\in[a,b]} |f(x)| = \|f\|$

Hence, $\|f\| > 0$ if $f \neq 0$.
We also need to show that $\|f\| = 0 \Rightarrow f = 0$.
We show this by contrapositive argument.
Assume $f(x) \neq 0$, then as above $\exists\ c \in [a,b]$ so that $f(c) \neq 0$, and $\|f\| > 0 \Rightarrow \|f\| \neq 0$.
Since $f \neq 0 \Rightarrow \|f\| \neq 0$, then by contrapositive argument, this is equivalent to $\|f\| = 0 \Rightarrow f(x) = 0$ in the function space $C[a,b]$.

2. Let $\alpha \in \mathbb{R}$, then

$$\|\alpha f\| = \max_{x\in[a,b]} |\alpha f(x)|$$

$$= \max_{x\in[a,b]} |\alpha||f(x)|$$

$$= |\alpha| \max_{x\in[a,b]} |f(x)|$$

$$= |\alpha| \|f\|$$

3. (Δ inequality) Let $f, g \in C[a,b]$. Then

$$\|f+g\| = \max_{x\in[a,b]} |f(x)+g(x)|$$

$$\leq \max_{x\in[a,b]} \Big[|f(x)| + |g(x)|\Big]$$

$$\leq \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)|$$

$$= \|f\| + \|g\|$$

So, the 'sup' norm is a norm on $C[a,b]$.

When we plan to use a given norm, $\|\cdot\|$, with some specific type of functions, the function space that we work with is defined by including all those functions that have a <u>finite norm</u>

<u>Example 1</u>

$L^2[a,b] = \{f(x) \mid f \text{ is defined for all } x\in[a,b] \text{ and } \|f\|_2 < \infty\}$

Here,

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}$$

L^2-norm is finite

<u>Example 2</u>

$C[0,1] = \{f(x) \mid f \text{ is continuous for all } x\in[0,1]\}$

We typically use the sup-norm with this set

$$\|f\| = \max_{x\in[a,b]} |f(x)|$$