Once we have seen the previous examples of Cauchy sequences that don't converge w.r.t. L^2-norm on $C[0,1]$, we begin to think about all the types of functions that must be added to $C[0,1]$ so that the new space will contain all of its limit "points" (or limit fctns here). Your textbook gives a nice discussion of what is needed. And the conclusion is:

"The space of all continuous functions with norm described by

$$\|f\|^2 = \int_0^1 |f(t)|^2 dt < \infty$$

cannot be a complete space."

Here, he is referring to the Riemann integral

* Mathematicians' "Dirty Little Secret" – when one is at an impasse of this type, attempt to "define" your way out of the situation. In this case, we redefine what we mean by $\int_0^1 |f(t)|^2 dt$

If we use the theory of Lebesgue Integration, then we essentially redefine what we mean by integration. With the new definition of integration, we can show that the space of functions with

$$\|f\|^2 = \underbrace{\int_0^1 |f(t)|^2 dt}_{\text{when this represents Lebesgue integration}} < \infty$$

is Complete!

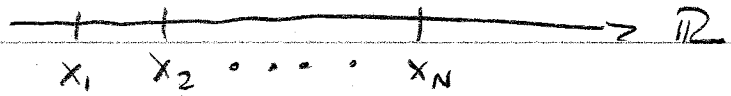
Note: this was a "game-changer" for Applied Mathematicians.

- Riemann Integration computes integrals via sums that look like $\Sigma f(x_i)\Delta x_i$

- Lebesgue integration computes integrals via considering sets, their "measure" + the way that functions are defined on those sets.

1. A set of real #s is a <u>set of measure zero</u> if for every $\varepsilon > 0$, the set can be "covered" by a collection of open intervals whose total length is less than ε.

<u>Ex:</u> A finite set of real #s has measure $= 0$.

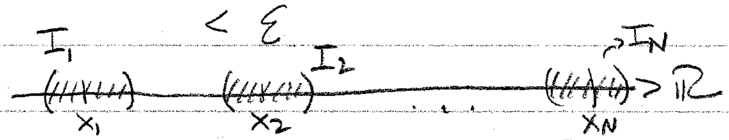


$S = \{x_1, x_2, \ldots, x_N\}$

For any $\varepsilon > 0$, we define $I_j = \left(x_j - \frac{\varepsilon}{2(N+1)}, x_j + \frac{\varepsilon}{2(N+1)}\right)$, $j = 1, 2, \ldots, N$
Then $x_j \in I_j$, so I_j "covers" x_j for each $j = 1, 2, \ldots, N$.
And the total length of the set of intervals is

$$\begin{aligned} \text{Total Length} &= N \cdot \text{length of } I_j \quad \leftarrow \text{because they are all the same length} \\ &= N\left[2\left(\frac{\varepsilon}{2(N+1)}\right)\right] \\ &= \frac{N\varepsilon}{N+1} \\ &< \varepsilon \end{aligned}$$

<u>Picture:</u>



Ex: Any countable set of real #s has measure zero (see your textbook for details)

For distinction purposes, $\int_R$ denote the Riemann integral + $\int_L$ denote the Lebesgue integral

2.) If $\int_R f(t)dt$ exists, (ie. finite), then $\int_L f(t)dt$ exists
AND
$$\int_L f(t)dt = \int_R f(t)dt$$

3.) If $\int_L f(t)dt$ + $\int_L g(t)dt$ exists and α is scalar, then
$$\int_L \alpha f(t)dt = \alpha \int_L f(t)dt \quad \text{\&} \quad \int_L (f+g)(t)dt = \int_L f(t)dt + \int_L g(t)dt$$

$\int_L$ satisfies properties of integrals that make it a linear operator.

4.) If $\int_L [f(t)]^2 dt$ and If $\int_L [g(t)]^2 dt$ exist, then
$$\int_L f(t)g(t)dt \quad \text{and} \quad \int_L [f+g]^2 dt \quad \text{both exist.}$$

→ these are important for the closure property of L^2 as an inner product space.

5.) If f and g are equal, except on a set of measure 0, then
$$\int_L (f-g)dt = 0 \quad \text{and} \quad \int_L [f-g]^2 dt = 0.$$

Property 5. on previous page means that if two fctns differ from each other only on a set of measure 0, then the L^2-distance, ie $\|f-g\| = \int_L f-g \, dt$, between the two functions is zero, ie. $\|f-g\|=0$ in the Lebesgue sense.

So, if we measure distance in the "uniform norm", then $\|f-g\|=0 \Rightarrow \max_{t\in[a,b]} |f(t)-g(t)| = 0$

$$\Rightarrow f(t)=g(t) \text{ for all } t\in[a,b].$$

But if we measure distance in the L^2-norm, then $\|f-g\|=0 \Rightarrow \left(\int_a^b |f(t)-g(t)|^2 dt\right)^{1/2} = 0$

$\Rightarrow f(t)=g(t)$, except possibly on a set of measure 0 in $[a,b]$.

- If $\|f-g\|=0$ in the Lebesgue sense, then we sometimes say that f and g are "<u>equivalent</u>"

<u>Other really important stuff:</u>

- $L^2[a,b] = \{ f \mid \underbrace{\left(\int_a^b |f(t)|^2 dt\right)^{1/2}}_{\text{Lebesgue integral here}} < \infty \}$ is Complete!

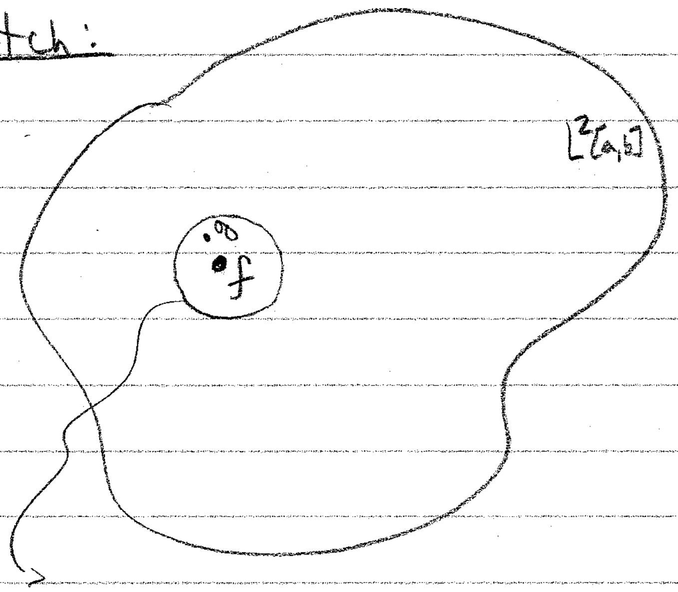
- Continuous Functions are <u>dense</u> in $L^2[a,b]$
This means that if we choose an element $f \in L^2[a,b]$, then any neighborhood of that element contains a continuous function.

By 'neighborhood', we mean to examine the set

$$N_\varepsilon = \{ g \in L^2[a,b] \mid \|f-g\| < \varepsilon \}$$

Usually we call this an "ε-neighborhood" OR "ε-ball" around the pt. f.

<u>Sketch:</u>



ε-Neighborhood of f
No matter how small this ball gets, we can always find a continuous function in the set.

So, the continuous functions (the 'nice' ones) are arbitrarily close to the weird / pathological functions in $L^2[a,b]$ AND vice versa!

2 thms are listed on p64-65
Lebesgue Dominated Convergence Theorem and
Fubini's Theorem
(You will have a chance to play with LDCT
on your homework)

Defn: An inner product space that is complete is
called a Hilbert Space.

[By 'complete' here, we mean w.r.t the norm
which is induced by the inner product -
completeness is w.r.t. that particular norm/inner product]

$\mathbb{R}^n$ is complete w.r.t any of its norms.

- If we consider $\mathbb{R}^n$ with $\|x\|_2 = \sqrt{\langle x,x\rangle} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$,
then this is a Hilbert Space.

- If we consider $\mathbb{R}^n$ with $\|x\| = \max_{i=1,2,..n} |x_i|$, then
this is a Banach space - because the norm is
NOT the one induced by the inner product -.

Ex: For sequence space ℓ^2, the inner product is
$\langle x,y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} \Rightarrow \|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$ is induced norm

ℓ^2 with this $\langle\cdot,\cdot\rangle$ is a Hilbert Space.

$L^2[a,b]$ is equipped with the inner product

$$\langle f,g\rangle = \int_a^b f(t)g(t)\,dt \quad \forall\, f,g \in L^2[a,b]$$

Property 4 on pg. 64 to see that this innerproduct is well-defined.

- Other comments with regard to infinite dimensional spaces, orthogonality, Gram-Schmidt & Cauchy-Schwarz all translate to these spaces as well. So,
  $$|\langle x,y\rangle| \le \|x\|\,\|y\| \text{ holds for}$$
  x, y in one of these inner product spaces.

- Triangle Inequality follows too!

Sobolev Spaces

- $L^2[a,b]$ is the space that we use to build Sobolev Spaces.

- Sobolev Spaces get used ALOT for the analysis, soln, approximation related to PDEs.

- They are subspaces of $L^2[a,b]$ that are also complete! We restrict attention to functions in $L^2[a,b]$ that have a certain amount of smoothness; ie a certain # of their derivatives are required to be in $L^2[a,b]$ as well.

Ex: $H^1[a,b] = \{ f \in L^2[a,b] \mid f' \in L^2[a,b] \text{ also} \}$

- Inner product on $H^1[a,b]$ is
$$\langle f,g \rangle_1 = \int_a^b f(x)\overline{g(x)} + f'(x)\overline{g'(x)}\,dx \longleftrightarrow \left\{ \begin{array}{l} \text{well-defined since } f, f' \\ \text{are both in } L^2[a,b] \end{array} \right\}$$

Note: Complex conjugation $\overline{g(x)}, \overline{g'(x)}$ is only needed if we allow $f(x), g(x)$ to take on complex values. In this class, we will usually drop the "conjugate" notation & only use it if needed!

- Induced Norm is
$$\| f \| = (\langle f,f \rangle)^{1/2} = \left(\int_a^b |f(x)|^2 + |f'(x)|^2\,dx \right)^{1/2}$$

So, the superscript 1 for $H^1[a,b]$ denotes the highest order derivative of $f(x)$ that is required to be L^2-integrable.

Similarly, $H^n[a,b] = \{f \in L^2[a,b] \mid \underbrace{f, f', \ldots, f^{(n)}}_{} \in L^2[a,b]\}$

derivatives up to order n
are L^2-integrable

- Inner Product on $H^n[a,b]$ is

$$\langle f,g\rangle_n = \int_a^b f(x)\overline{g(x)} + f'(x)\overline{g'(x)} + \cdots + f^{(n)}(x)\overline{g^{(n)}(x)}\, dx$$

$$= \int_a^b \sum_{j=0}^{n} \frac{d^j}{dx^j} f(x) \cdot \overline{\frac{d^j}{dx^j} g(x)}\, dx$$

- Induced Norm is

$$\| f \| = \sqrt{\langle f,g\rangle_n} = \left(\int_a^b \sum_{j=0}^{n} \left| \frac{d^j}{dx^j} f(x) \right|^2 dx \right)^{1/2}$$

<u>Important things to Know:</u>

1. Sobolev Spaces are complete!

2. <u>Sobolew Inequalities</u>

<u>Thm 2.1</u>

1. If $f \in H^1[a,b]$, then f is equivalent to a continuous function, and if M is any closed & bounded set containing x, then there is a constant $C \geq 0$ independent of f, so that
$$|f(x)| \leq C \left(\int_M |f(x)|^2 + |f'(x)|^2 dx \right)^{1/2} \quad \text{for all } x \in M.$$

2. If $f \in H^{k+1}[a,b]$, then f is equivalent to a C^k-function, i.e. a function whose derivatives up to order k are continuous.