Section 3.2 Bounded (Continuous) Linear Operators in Hilbert Space

Recall that we have often characterized a matrix $A \in \mathbb{R}^{n \times m}$ as a Linear operator mapping $\mathbb{R}^m \to \mathbb{R}^n$. Now that we have laid all the mathematical groundwork, it is time to consider other types of operators - such as differential and integral operators. These guys will map a Hilbert space to (another, sometimes) Hilbert space. We will see similarities between the Linear algebra results in Chapter 1 and the results of this discussion of Linear Operators.

Defn 3.1 A Bounded (OR Continuous) Linear Operator $L: H \to H$ is a mapping which takes one function in in H to another function in H and satisfies the two properties

(1.) L is Linear: $L(\alpha f + \beta g) = \alpha Lf + \beta Lg$ for all $\alpha, \beta \in \mathbb{R}$ and for all $f, g \in H$.

(2.) L is Bounded (OR continuous): There is a constant $C > 0$ so that

$$\|Lf\| \leq C \|f\| \quad \text{for all } f \in H$$

Remarks:
- H denotes a Hilbert Space
- $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ denote the inner-product and induced norm on H.
- Linear Operators can also map one Hibert space H to another Hilbert space $\tilde{H}$.

Defn: The Norm of the operator L is defined as

$$\|L\| = \underbrace{\sup_{u\neq 0} \frac{\|Lu\|}{\|u\|}}_{\text{Defn given in textbook}} = \underbrace{\sup_{\|u\|=1} \|Lu\|}_{\text{this is an equivalent defn. that is sometimes useful.}}$$

Remark: $\|L\|$ is the smallest constant C for which $\|Lu\| \leq C\|u\|$ for all $u \in H$.

Examples:

① Identity Operator $Lf = f \quad \forall f \in H$

Clearly Linear: $L(\alpha f + \beta g) = \alpha f + \beta g$ by defn of L

$= \alpha Lf + \beta Lg$ by defn of L

Clearly Bdd: $\|Lf\| = \|f\| \quad \forall f \in H$, so $C = 1$ for Defn. 3.1

Also, $\|L\| = 1$ since $\|L\| = \sup_{u\neq 0} \frac{\|Lu\|}{\|u\|} = \sup_{u \neq 0} \frac{\|u\|}{\|u\|} = 1$

② Let $H = L^2[a,b]$ and let $L: H \to \mathbb{R}$ by

$$Lf = \int_a^b f(x)\,dx \quad \forall f \in H$$

Clearly Linear: $L(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)(x)\,dx$

$$= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

$= \alpha Lf + \beta Lg \quad \forall f, g \in H$ and $\alpha, \beta \in \mathbb{R}$

For Bdd-ness: Begin by looking at $\|Lf\|^2$
For any $f \in H$,

$$\|Lf\|^2 = |Lf|^2 \qquad \text{since } Lf \in \mathbb{R},\ \|Lf\| = |Lf|$$

$$= \left| \int_a^b f(x)\,dx \right|^2$$

$$= \underbrace{\left| \langle f, 1 \rangle \right|^2}_{L^2\text{-innerproduct of } f(x) \text{ and the constant function } 1}$$

$$\le \left(\|f\|_{L^2} \|1\|_{L^2} \right)^2 \qquad \text{by Schwarz Inequality}$$

$$= \left(\left(\int_a^b [f(x)]^2 dx \right)^{1/2} \left(\int_a^b 1^2 dx \right)^{1/2} \right)^2$$

$$= \int_a^b [f(x)]^2 dx \,(b-a)$$

$$= \|f\|_{L^2}^2 (b-a)$$

So $|Lf|^2 \le (b-a) \|f\|^2$

and $|Lf| \le \sqrt{b-a}\, \|f\| \quad \forall f \in H$, and L is Bdd.

- This means that the operator norm $\|L\| \le \sqrt{b-a}$

③ An Integral Operator Example
Let $k(x,y)$ be a function satisfying

$$\int_a^b \int_a^b [k(x,y)]^2 \,dx\,dy < \infty$$

- Think of $k(x,y)$ as our "kernel" from Section 3.1

Let $H = L^2[a,b]$, and define $K : H \to H$ by

$$Ku = \int_a^b k(x,y)\,u(y)\,dy, \quad a \le x \le b$$

So Ku is a function of x that is defined for all $x \in [a,b]$

<u>K is Linear</u>: Let $\alpha, \beta \in \mathbb{R}$, $u, v \in L^2[a,b]$. Then

$$K(\alpha u + \beta v) = \int_a^b k(x,y)[\alpha u + \beta v](y)\,dy$$

$$= \alpha \int_a^b k(x,y)\,u(y)\,dy + \beta \int_a^b k(x,y)\,v(y)\,dy$$

$$= \alpha Ku + \beta Kv \qquad \forall \alpha, \beta \in \mathbb{R},\ u, v \in H$$

<u>K is Bounded</u>:

$$\|Ku\|^2 = \int_a^b (Ku)^2\,dx$$

$$= \int_a^b \left(\int_a^b k(x,y)\,u(y)\,dy\right)^2 dx$$

$$= \int_a^b \langle k, u\rangle^2\,dx \qquad \overset{L^2\text{-innerproduct}}{\langle k, u\rangle = \int_a^b k(x,y)\,u(y)\,dy}$$

$$\le \int_a^b \|k\|_{L^2}^2\,\|u\|_{L^2}^2\,dx \quad \text{by Cauchy-Schwarz Ineq.}$$

$$= \int_a^b \left(\int_a^b [k(x,y)]^2\,dy\right)\left(\int_a^b [u(y)]^2\,dy\right) dx$$

$$= \|u\|_{L^2}^2 \underbrace{\int_a^b \int_a^b [k(x,y)]^2\,dy\,dx}_{< \infty \text{ by our assumption on page 3}}$$

Hence, if we let $C = \int_a^b \int_a^b [k(x,y)]^2 \, dx\,dy$, then

$$\|Ku\|_{L^2}^2 \leq C \|u\|_{L^2}^2$$

$$\Rightarrow \quad \|Ku\| \leq \sqrt{C} \, \|u\| \qquad \forall u \in H.$$

And K is a Bounded Linear operator.//

- Note that the above argument also shows that the operator norm
$$\|K\| \leq \sqrt{C}$$

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(4) Let $g(x) \in L^2[a,b]$. Once we choose this fixed function $g \in L^2[a,b]$, we can always define an associated Bdd Linear operator $L_g : L^2[a,b] \longrightarrow \mathbb{R}$ by

$$L_g f = L_g(f) = \int_a^b g(x) f(x) \, dx = \langle g, f \rangle$$

In a similar fashion to the previous arguments, we can show that

L_g is Linear: Let $\alpha, \beta \in \mathbb{R}$ and $f, h \in L^2[a,b]$

$$L_g(\alpha f + \beta h) = \int_a^b g(x)[\alpha f + \beta h](x) \, dx$$

$$= \alpha \int_a^b g(x) f(x) \, dx + \beta \int_a^b g(x) h(x) \, dx$$

$$= \alpha L_g f + \beta L_g h \qquad \forall \alpha, \beta \in \mathbb{R} \text{ and } f, h \in L^2[a,b]$$

So, L_g is a Linear Operator.

L_g is Bounded:

$$\|L_g f\| = |L_g f| \quad \text{since } L_g f \in \mathbb{R}$$

$$= \left| \int_a^b g(x) f(x)\, dx \right|$$

$$= |\langle g, f \rangle| \rightarrow L^2\text{-inner product}$$

$$\leq \underbrace{\|g\| \, \|f\|}_{L^2\text{-norms}} \quad \text{By Schwarz Ineq.}$$

Since $g(x)$ is a fixed function in $L^2[a,b]$, $\|g\| = C$ for some finite constant $C > 0$. Hence

$$\|L_g f\| \leq \|g\| \, \|f\| = C \|f\|$$

And, L_g is a Bounded Linear operator. //

- In addition, as an operator, $\|L_g\| \leq \|g\|$.

⑤ One can generalize the previous example to any Hilbert Space. Let $w \in H$. Then for this fixed w, one can associate the Linear operator $L_w : H \to \mathbb{R}$ by

$$L_w v = \langle v, w \rangle \quad \text{for all } v \in H$$

→ the inner-product associated w/ H.

By Properties of $\langle \cdot, \cdot \rangle$ and by Schwarz inequality, it can be shown that L_w is a Bounded Linear operator with $\|L_w\| \leq \|w\|$

One Example of an UNBOUNDED Linear operator

Differential operators provide us with classic examples of linear operators that are NOT BOUNDED.

Let $L : H^1[a,b] \longrightarrow L^2[a,b]$ by $Lf = \frac{df}{dx}$

Clearly L is Linear: Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in H^1[a,b]$.
Then $L(\alpha f + \beta g) = \frac{d}{dx}(\alpha f + \beta g)(x)$

$$= \alpha \frac{df}{dx} + \beta \frac{dg}{dx}$$

$$= \alpha Lf + \beta Lg \qquad \forall \alpha, \beta \in \mathbb{R} \text{ and } f, g \in H^1[a,b]$$

L is Unbounded
Choose the sequence $f_n(x) = \sin(nx)$ with $a = 0$, $b = 2\pi$.
Then

$$Lf_n = n\cos(nx) \qquad \forall\ n = 1, 2, \ldots$$

and

$$\|Lf_n\|^2 = \int_0^{2\pi} (n^2 \cos^2(nx))\,dx = n^2 \int_0^{2\pi} \cos^2(nx)\,dx = n^2\pi$$

But $\|f_n\|^2 = \int_0^{2\pi} \sin^2(nx)\,dx = \pi$. If we put these together, we see that

$$\|Lf_n\|^2 = n^2\pi = n^2 \|f_n\|^2$$

$\Rightarrow \quad \|Lf_n\| = n\|f_n\|$ for all n.

Hence, there is NO constant C for which $\|Lf\| \le C\|f\|$ $\forall f \in H^1$ and L is not a BOUNDED operator.