The Rest of Section 3.2: L^* and FAT for L^*

- Bdd Linear Operators (BLOs) have adjoints!

Theorem 3.2

If $L: H \to H$ is a bounded linear operator,
then
1. L^* exists
2. L^* is also a Bounded Linear Operator

Pf: First, we can use the BLO, here L, to construct a bdd. linear functional $T: H \to \mathbb{R}$. Define T in the following way. Fix an element $v \in H$. And define
$$Tu = \langle Lu, v \rangle \quad \forall u \in H$$
(So the element v that one chooses in H determines T also!)

– T is a Linear functional since L is a Linear operator:
$$\begin{aligned} T(\alpha u_1 + \beta u_2) &= \langle L(\alpha u_1 + \beta u_2), v \rangle \\ &= \langle \alpha L u_1 + \beta L u_2, v \rangle && \text{by Linearity of } L \\ &= \alpha \langle L u_1, v \rangle + \beta \langle L u_2, v \rangle && \text{by properties of } \langle \cdot, \cdot \rangle \\ &= \alpha T u_1 + \beta T u_2 && \text{by defn. of } T \end{aligned}$$

– T is a bdd Linear functional since L is a B.L.O.:
$$\begin{aligned} |Tu| &= |\langle Lu, v \rangle| && \text{by defn. of } T \\ &\le \|Lu\| \|v\| && \text{by Cauchy-Schwarz Ineq.} \\ &\le C\|u\| \|v\| && \text{since } L \text{ is B.L.O.} \\ &= \underbrace{(C\|v\|)}_{\text{a finite constant for our fixed } v \in H.}\|u\| \end{aligned}$$
So, T is Bdd. Lin. Fctnal.

- So, T is a Bdd linear functional on H. Then by <u>Riesz Representation Theorem</u>, there exists! $g \in H$ so that $Tu = \langle u, g \rangle \ \forall u \in H$.

One we have determined g, we use it to construct the adjoint operator L^*. Then for our fixed $v \in H$, we define

$$L^* v = g$$

And if we do this for each $v \in H$, then this defines the action of L^* on H so that

$$\begin{aligned} \langle Lu, v \rangle &= Tu && \text{by our construction of } T \\ &= \langle u, g \rangle && \text{that came from Riesz Rep. Thm} \\ &= \langle u, L^* v \rangle && \text{by our construction of } L^* \end{aligned}$$

(So, our construction satisfies the defn of the adjoint L^*)
By construction, L^* exists, and 1 of Thm is shown.

- <u>L^* is Linear</u>:
Assume $L^* v_1 = g_1$ and $L^* v_2 = g_2$ for $v_1, v_2, g_1, g_2 \in H$, $\underbrace{\alpha, \beta \in \mathbb{C}}$
And spse $L^*(\alpha v_1 + \beta v_2) = g_3$ for some $g_3 \in H$.

Yes, he seems to pay attention to $\mathbb{C}$ here?

Then

$$\begin{aligned} \langle u, g_3 \rangle &= \langle u, L^*(\alpha v_1 + \beta v_2) \rangle && \forall u \in H \\ &= \langle Lu, \alpha v_1 + \beta v_2 \rangle \\ &= \bar{\alpha} \langle Lu, v_1 \rangle + \bar{\beta} \langle Lu, v_2 \rangle \\ &= \bar{\alpha} \langle u, L^* v_1 \rangle + \bar{\beta} \langle u, L^* v_2 \rangle \\ &= \langle u, \alpha L^* v_1 \rangle + \langle u, \beta L^* v_2 \rangle \\ &= \langle u, \alpha L^* v_1 + \beta L^* v_2 \rangle && \forall u \in H \\ &= \langle u, \alpha g_1 + \beta g_2 \rangle && \forall u \in H \end{aligned}$$

So, $\langle u, g_3 - [\alpha g_1 + \beta g_2] \rangle = 0 \quad \forall u \in H$.

In particular, this is true for $u = g_3 - [\alpha g_1 + \beta g_2] \in H$

So

$$\langle g_3 - [\alpha g_1 + \beta g_2], g_3 - [\alpha g_1 + \beta g_2] \rangle = 0$$

$\Rightarrow \quad \| g_3 - [\alpha g_1 + \beta g_2] \|^2 = 0$ by defn of $\| \; \| = \sqrt{\langle \cdot, \cdot \rangle}$

$\Rightarrow \quad \| g_3 - [\alpha g_1 + \beta g_2] \| = 0$

$\Rightarrow \quad g_3 - [\alpha g_1 + \beta g_2] = 0 \in H$ by property of $\| \cdot \|$

$$g_3 = \alpha g_1 + \beta g_2 \in H$$

And this means

$$L^*(\alpha v_1 + \beta v_2) = \alpha L^* v_1 + \beta L^* v_2 \quad \forall v_1, v_2 \in H, \alpha, \beta \in \mathbb{C}$$

Hence L^* is a Linear Operator.

- <u>L is Bounded:</u> Let $v \in H$, then

$$\begin{aligned}
\|L^* v\|^2 &= \langle L^* v, L^* v \rangle \\
&= \langle L(L^* v), v \rangle \\
&\leq |\langle L(L^* v), v \rangle| \\
&\leq \| L(L^* v) \| \, \|v\| \quad \text{by Cauchy-Schwarz Ineq.} \\
&\leq \underbrace{C \, \|L^* v\|}_{\text{Since } L \text{ is a B.L.O., } C>0 \text{ a finite constant}} \|v\|
\end{aligned}$$

So $\| L^* v \|^2 \leq C \|L^* v\| \, \|v\|$

If $\|L^* v\| \neq 0$, then $\| L^* v \| \leq C \|v\|$ from above.

And note that if $\|L^* v\| = 0$, then

$$0 = \|L^* v\| \leq C \|v\| \quad \text{for all } v \in H$$

So, we see that $\|L^* v\| \leq C \|v\|$ and L^* is Bounded.

Hence, 2. of Thm 3.2 is shown, and we are done. //

FAT (again!)

Theorem 3.3 Fredholm Alternative Theorem (FAT)
If L is a Bounded Linear operator in H with closed range, then the equation $Lf = g$ has a solution if and only if $\langle g, v\rangle = 0 \ \forall \ v \in N(L^*)$.

Consequences:
Denote $\overline{R(L)}$ = closure of Range of L operator.
Then FAT tells us that
$$H = \overline{R(L)} \oplus N(L^*)$$
So, for any $g \in H$, there is a unique representation

$$g = g_r + g_n$$

with $g_r \in \overline{R(L)}$ and $g_n \in N(L^*)$. And this decomposition is orthogonal. That is, $\langle g_r, g_n\rangle = 0$.

These comments should remind us of our results in Chapter 1 when $A \in \mathbb{R}^{m\times n}$, and we wrote
$$\mathbb{R}^m = R(A) \oplus N(A^T)$$
We used often to understand solns of $Ax = b$ for A not square. Thm 3.3 will have a similar usefulness for us in the coming sections.
