Chapter 3: Integral Equations

Integral EQNs arise in various settings. They can often provide alternative formulations of B.V.P.s. Some advantages of integral equations are:

1. The integral operator appearing in the eqn is a bounded operator, and often compact, whereas the differential operator is unbounded. This has very important implications when we attempt to construct sequences of approximations to solns of these equations.

2. The bdry conditions are incorporated into the integral equation through its kernel, which is a Green's Fctn. (more later on this)

3. Associated with the integral eqn are variational principles and approximation schemes that complement those arising from the formulation as a differential equation. These ideas are similar to that of Finite Element schemes that we just covered.

4. Some B.V.P.s for PDEs can be translated into integral equations of lower dimensionality. (Not sure if we will see much of this in this class)

Examples

- Population Dynamics model

  t = time

  $u(t)$ = # of individuals in the population at time t

  A = growth/death rate is a fixed constant.

  $f(t)$ = net rate of influx of individuals due to migration

  u_0 = initial population

ODE:

$$\frac{du}{dt} = Au + f(t) \quad , u(0) = u_0$$

If we use an integrating factor of $\mu(t) = e^{-At}$, then we can solve for $u(t)$. (Note: If you needs notes to remind you of an "integrating factor", ask me)

$$u(t) = u_0 e^{At} + \int_0^t e^{A(t-\tau)} f(\tau)\, d\tau$$

An integral equation: given $f(t)$ and u_0, find $u(t)$.

Fredholm Integral EQN of the 1st Kind:

$$u(t) = \int_a^b k(t,\tau) f(\tau)\, d\tau \qquad (*)$$

- $k(t,\tau)$ is called the "kernel" of the equation.

Volterra Integral EQN:

Special Case for $(*)$ when $k(t,\tau) = 0$ for $\tau > t$

If we consider the population dynamics ODE

$$\frac{du}{dt} - Au = f \ , \ u(0) = u_0 .$$

then we can define an associated operator known as a differential operator L according to the expression

$$Lu = \frac{du}{dt} - Au$$

← describes the operator by its action on an arbitrary input function $u(t)$

- Sometimes we use the shorthand notation $L = \frac{d}{dt} - A$

- The representation

$$u(t) = u_0 e^{At} + \int_0^t e^{A(t-\tau)} f(\tau) d\tau$$

gives us a form for the inverse of the L-operator. That is, given a right-hand side function $f(t)$ and an initial condition u_0, we can determine $u(t)$ using L^{-1} -> the inverse operator for L. this is defined by

$$L^{-1}(f(t), u_0) = u_0 e^{At} + \int_0^t e^{A(t-\tau)} f(\tau) d\tau = u(t)$$

- We will re-visit the issue of inverses of operators later.

- <u>Fredholm Integral EQN of the 2nd Kind</u>:

$$u(t) = \int_a^b k(t,\tau) u(\tau) d\tau + f(t)$$

→ $f(t)$ is some given function

unknown is now inside the integral also

- $k(t,\tau)$ is still our kernel → usually given to us.

Differential EQNS can be converted to Integral EQNS

$$\frac{d^2u}{dx^2} = f(x)$$
$$u(0)=0,\ u(1)=0 \quad \Big] \text{ BVP}$$

- Using Integration By Parts, one can show that the solution of the B.V.P. is given by

(3.1) $$u(x) = \int_0^1 k(x,y) f(y)\,dy$$ ← Fredholm of 1^{st} Kind.

Where

$$k(x,y) = \begin{cases} y(x-1), & 0 \le y < x \le 1 \\ x(y-1), & 0 \le x < y \le 1 \end{cases}$$

- If we take this a step further & allow $f = f(x,u)$, then the new B.V.P.

$$\frac{d^2u}{dx^2} = -\omega(x)u + g(x).$$
$$u(0)=0,\ u(1)=0$$

is equivalent to

$$u(x) = \int_0^1 k(x,y) g(y)\,dy - \int_0^1 k(x,y)\omega(y)\underbrace{u(y)}\,dy$$

- Fredholm Integral EQN of the 2^{nd} Kind → Note that $u(y)$ appears in the integral
- Also note the constant limits of integration

- One of your last assignments will be to verify eqn (3.1). You will need Leibniz Rule for this. (Ask me if you haven't seen it OR don't remember it.)

Example: (Also from your text) Population Renewal Process

t = time

(unknown) $B(t)$ = rate of births at time t in a population.

(known) $f(t)$ = rate of births from individuals who were born elsewhere & have migrated

- This model takes into account the age structure of the population. The current births are only from individuals born previously who are now bearing offspring

$B(t-a)$ = rate of births a units of time in the past

$s(a)$ = probability density function describing survival to age a. Assume $s(a)=0$ for $a > a_\infty$

$\beta(a)$ = fertility of individuals who are a units of age

$$B(t) = \int_0^{a_\infty} \beta(a)s(a)\underbrace{B(t-a)}_{\text{unknown sits inside integral also}}da \; + f(t)$$

- Limits of integration are known constants.

- Using a change of variable: $\tau = t - a$ $\quad$ (t is fixed here)

$$d\tau = -da$$

$$B(\tau+a) = \int_t^{t-a_\infty} \beta(t-\tau)s(t-\tau)B(\tau)(-d\tau) \; + f(\tau+a)$$

OR

$$B(t) = \int_{t-a_\infty}^{t} \beta(t-\tau)s(t-\tau)\underbrace{B(\tau)}d\tau \; + f(t)$$

- Volterra Integral EQN: t is in limits of integration & unknown sits inside integral
- Kernel is formed by $k(t-\tau) = \beta(t-\tau)s(t-\tau)$

- Note that the previous example gives an example where the independent variable t, appears in the limits of the integral. This is a Volterra Integral, not a Fredholm integral as the textbook state. (and this is NOT in the errata)

Volterra Integral Eqn

$$u(t) = \int_a^t k(t,\tau)\, u(\tau)\, d\tau + f(t)$$

Here, t is the independent variable for the unknown $u(t)$, and it appears explicitly in the limits of integration

For Fredholm & Volterra, we have the assumption that the kernel is L^2-integrable; i.e.,

$$\int_a^b \int_a^b [k(x,y)]^2\, dx\, dy < \infty$$

↳ These are called Hilbert-Schmidt kernels.

Another class of integral equations:

Singular Integral Equations

- (Here, the kernel $k(x,y)$ has a singularity along the line $y=x$)

- Your textbook gives a nice example of this type relating to an application of X-RAY TOMOGRAPHY. I think it's pretty interesting!

In 1963, A.M. Cormack used the following mathematical ideas to help him build the first CAT SCAN (Computer Aided Tomography – CAT) Nifty, huh!

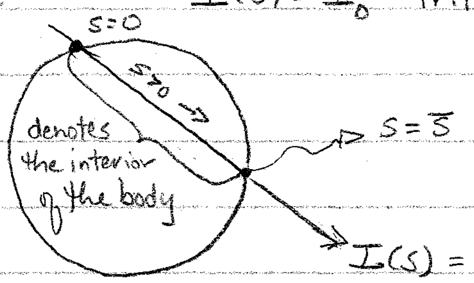
(A1) Spse a high frequency electromagnetic wave penetrates an object – modelled by a sphere for us. Spse the wave travels along a straight line without bending or spreading. Spse the intensity I decreases as it travels (because of absorption).
(A2) Assume the x-ray is absorbed at a linear rate $g = g(s)$ which is an unknown function of position.

s = position along the straight line
$I(s)$ = Intensity of wave (unknown)
$g(s)$ = Absorption rate of object (also unknown)
I_0 = initial intensity of x-ray

- A bit of background: (Taken from Stakgold, pg. 373) As an electromagnetic wave travels along the straight line through the body, it is absorbed at different rates by healthy and diseased tissues. By observing waves emanating from different directions as they have passed through the body, it may be possible to detect if and where tumors exist in the body.

- $g(s)$ may be referred to as an "absorption coefficient"

Picture: $I(0) = I_0$ = initial intensity of X-RAY



$I(s)$ = intensity of wave along this line, denote the Line by L

(A2) gives us that $g(s)$, $I(s)$ satisfy the relation

$$\frac{dI}{ds} = -g(s) I \ , \ I(0) = I_0$$

"Separation of Var's."

$$\frac{1}{I}\frac{dI}{ds} = -g(s)$$

$$\int_0^{\bar{s}} \frac{1}{I}\frac{dI}{ds}\, ds = -\int_0^{\bar{s}} g(s)\,ds$$

$$\int_{I_0}^{I(\bar{s})} \frac{1}{I}\, dI = -\int_0^{\bar{s}} g(s)\,ds$$

$$\ln(I) - \ln(I_0) = -\int_0^{\bar{s}} g(s)\,ds$$

$$\ln\left(\frac{I(\bar{s})}{I_0}\right) = -\int_0^{\bar{s}} g(s)\,ds$$

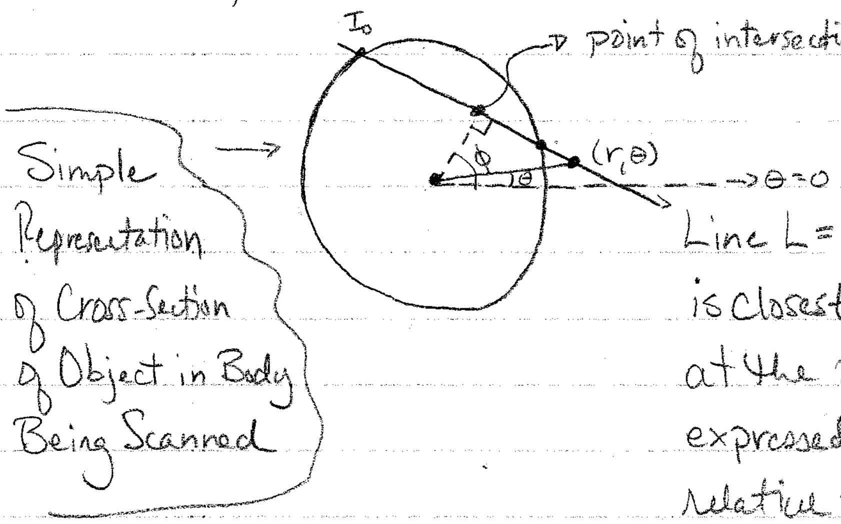
your book has the fraction backwards on pg. 103

$$\ln\left(I_0/I(\bar{s})\right) = \underbrace{\int_0^{\bar{s}} g(s)\,ds}$$

denotes a Line integral along the Line L.

- So, theoretically, one can measure $\ln\left(I_0/I(\bar{s})\right)$, and we want to find $g(s)$.
(Prepare yourself for polar coordinates, trig. and Complex Fourier Series)

Draw some pictures and convert $\int_L g(s)ds$ to polar coordinates

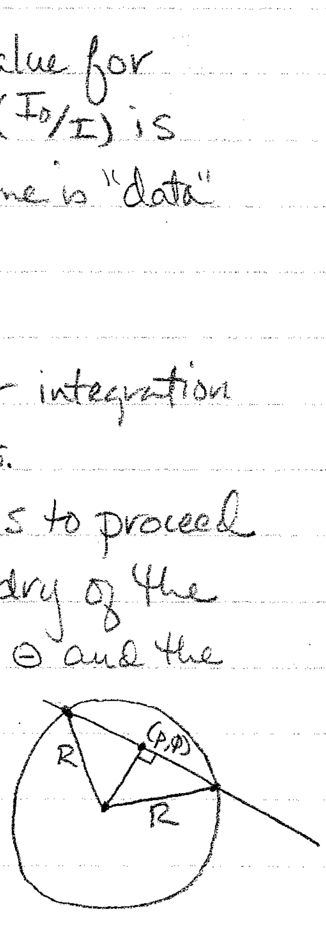


Line $L = L(\rho, \phi)$ is the line that is closest to the origin of the circle at the point (ρ, ϕ). Here, (ρ, ϕ) is expressed in polar coordinates relative to the origin of the circle

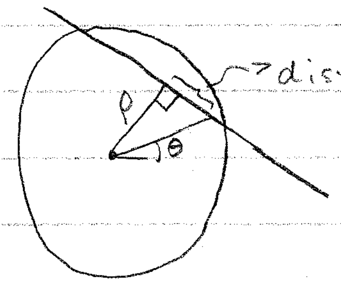
- Once we have converted to polar coordinates relative to our circle orientation, we define
$$f(\rho, \phi) = \int_{L(\rho,\phi)} g(r, \theta) ds = \ln\left(\frac{I_0}{I}\right)$$

- Note that as (ρ, ϕ) change, then the value for $\ln\left(\frac{I_0}{I}\right)$ will also change. But $\ln\left(\frac{I_0}{I}\right)$ is a measured quantity that we assume is "data" that is given to us.

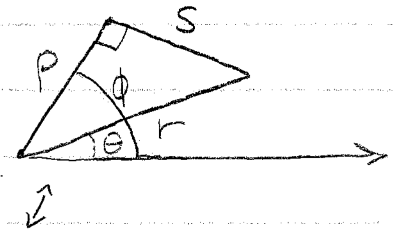
- Now, we have to adjust our ds in our integration to account for the change of variables. From the pt. (ρ, ϕ), there are 2 directions to proceed along the arc $L(\rho, \phi)$ to get to the bdry of the circle. One direction increases the angle θ and the other direction decreases the angle θ.



More pictures:



Bigger Picture



So, along the path of integration, the arclength is

$$s^2 + \rho^2 = r^2$$

- Using this, we can differentiate (recalling ρ^2 is fixed) to obtain

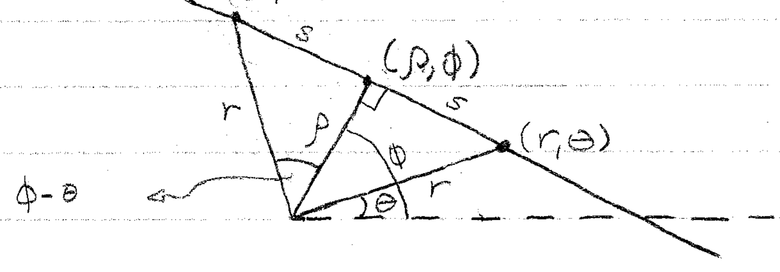
$$2s\,ds = 2r\,dr$$

$$ds = \frac{r\,dr}{s} = \frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

- One can also see from the Δ above that

$$\cos(\phi-\theta) = \frac{\rho}{r}$$

- The previous formula is for increasing s from the pt. (ρ, ϕ). Now, let's look at the case where we decrease s from the point (ρ,ϕ). The Bigger Picture Looks Like



So, the total angle is $\phi + \phi - \theta = 2\phi - \theta$

- Now, we have all of our Calculus machinery ready to compute some approximations

- Use Complex Fourier Series Expansion to approximate

$$g(r,\theta) = \sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta}$$

with the Fourier Coefficients given by

$$g_n(r) = \frac{1}{2\pi} \int_0^{2\pi} g(r,\theta) e^{-in\theta} d\theta$$

(Note: We didn't cover the complex version of this in Chapter 2, but it follows in a manner analogous to what we covered)

From $\ln(I_0/I) = \int_L g(r,\theta) ds$

$$= \int_\rho^R [g(r,\theta) + g(r, 2\phi-\theta)] ds$$

$$= \int_\rho^R \left[\sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta} + \sum_{n=-\infty}^{\infty} g_n(r) e^{in(2\phi-\theta)}\right] \frac{rdr}{\sqrt{r^2-\rho^2}}$$

$$= \int_\rho^R \sum_{n=-\infty}^{\infty} g_n(r) \left[e^{in\theta} + e^{in(2\phi-\theta)}\right] \frac{rdr}{\sqrt{r^2-\rho^2}}$$

$$= \int_\rho^R \sum_{n=-\infty}^{\infty} g_n(r) e^{in\phi} \left[e^{in(\theta-\phi)} + e^{in(\phi-\theta)}\right] \frac{rdr}{\sqrt{r^2-\rho^2}}$$

$$= \int_\rho^R \sum_{n=-\infty}^{\infty} g_n(r) e^{in\phi} \left[e^{-in(\phi-\theta)} + e^{in(\phi-\theta)}\right] \frac{rdr}{\sqrt{r^2-\rho^2}}$$

Recall that $e^{-i\beta} + e^{i\beta} = [\cos(-\beta) + i\sin(-\beta)] + [\cos(\beta) + i\sin(\beta)]$
$= 2\cos(\beta)$

So that the integrand simplifies to

$$\ln(I_o/I) = 2\int_{\rho}^{R} \sum_{n=-\infty}^{\infty} g_n(r) e^{in\phi} \cos(n(\phi-\theta)) \frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

$$= 2\sum_{n=-\infty}^{\infty} \int_{\rho}^{R} g_n(r) e^{in\phi} \cos(n(\phi-\theta)) \frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

Now, we also use a Complex Fourier Series expansion & define

$$\ln(I_o/I) = f(\rho,\phi) = \sum_{n=-\infty}^{\infty} f_n(\rho) e^{in\phi}$$

So that $f_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho,\phi) e^{-in\phi}\,d\phi$ by virtue of Fourier Series Results.

then

$$f(\rho,\phi) = \ln(I_o/I) = 2\sum_{n=-\infty}^{\infty} \int_{\rho}^{R} g_n(r) e^{in\phi} \cos(n(\phi-\theta)) \frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

By Defn

$$\sum_{n=-\infty}^{\infty} f_n(\rho) e^{in\phi} = \sum_{n=-\infty}^{\infty} \left[2\int_{\rho}^{R} g_n(r) \cos(n(\phi-\theta)) \frac{r\,dr}{\sqrt{r^2-\rho^2}} \right] e^{in\phi}$$

$\Rightarrow$

$$f_n(\rho) = 2\int_{\rho}^{R} g_n(r) \cos(n(\phi-\theta)) \frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

(Assumes the series converge)

And if we recall that $\cos(\phi-\theta) = \rho/r$ (from pg. 10), then we can use $\phi-\theta = \cos^{-1}(\rho/r)$ and substitute $\rightarrow$

$$f_n(\rho) = 2\int_\rho^R g_n(r)\cos\left(n\cos^{-1}(\rho/r)\right)\frac{r\,dr}{\sqrt{r^2-\rho^2}}$$

From Chapter 2, the Chebyshev Polynomials can be expressed as

$$T_n(x) = \cos\left(n\cos^{-1}(x)\right), \text{ for } n = 0,1,2,\ldots$$

and we use this to write

$$f_n(\rho) = 2\int_\rho^R \frac{g_n(r)\,T_n(\rho/r)}{\sqrt{r^2-\rho^2}}\,r\,dr$$

In principle, $f_n(\rho)$ is known (OR can be measured), and the goal is to find $g_n(r)$. This gives us an entire sequence of Singular Integral Equations to solve! (Of course, we're not going to be solving these in the near future, but it's an interesting example of applied math calculations)

- Remark: In your textbook, Keener assumes that our reference circle has radius $R=1$ for simplicity. This is easily achieved with some re-scaling, so our use of R in the upper limit of the integrals is acceptable