More Section 3.2 Linear Operators & Adjoints

- Linear Operators on Hilbert Spaces are the natural generalization of matrices in $\mathbb{R}^n$.

Let X and Y denote Hilbert Spaces, and let $L: X \to Y$ be a bdd. linear operator mapping $X \to Y$

Defns: $D(L) = \{ f \in X \mid Lf \text{ is defined} \}$ "Domain of L"

$R(L) = \{ y \in Y \mid Lf = y \text{ for some } f \in X \}$ "Range of L"

$N(L) = \{ f \in X \mid Lf = 0 \in Y \}$ "Null space of L"

Defn 3.2 Assume $L: H \to H$. Then the adjoint of the operator L is that operator L^* (if it exists) for which $\langle Lf, g \rangle = \langle f, L^* g \rangle \quad \forall f, g \in H$

(Note that there is a more general defn. for L^*, but we haven't laid all the groundwork for that defn. yet)

Ex: Recall the bdd, linear operator $Ku(x) = \int_a^b \int_a^b k(x,y) u(y) dy$
(Note, assume that all the integrals $\iint [k(x,y)]^2 dx dy$, $\int k(x,y) u(y) dy$, $\int k(x,y) v(x) dx$ exist & are finite)

We can compute the adjoint operator here →

Let $u, v \in H = L^2[a,b]$

$$\langle Ku, v\rangle = \int_a^b (Ku(x))\, v(x)\,dx$$

$$= \int_a^b \left[\int_a^b k(x,y)u(y)\,dy\right] v(x)\,dx$$

$$= \int_a^b \int_a^b k(x,y)u(y)\,v(x)\,dx\,dy$$ → Applying Fubini's Thm, pg. 64-65

$$= \int_a^b u(y)\left[\int_a^b k(x,y)v(x)\,dx\right] dy$$

$$= \int_a^b u(x)\left[\int_a^b k(y,x)\,v(y)\,dy\right] dx$$ Renaming Dummy variables of integration

$$= \langle u, K^* v\rangle$$

Where $K^* v(x) = \int_a^b k(y,x)\,v(y)\,dy \quad \forall \quad v \in H$

So, we have derived the action of the K^* operator.

Not all operators have adjoints — as they have been previously defined. We have to generalize our defn. of an adjoint to further investigate!

Ex: Let $Lf = \frac{df}{dx}$, where $L: H^1[0,1] \to L^2[0,1]$ with $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$ as the inner product. For an adjoint calculation, we examine the inner product →

$$\langle Lf, g\rangle = \int_0^1 [Lf]\, g(x)\,dx$$

$$= \int_0^1 f'(x)\, g(x)\,dx$$

I.B.P.

$$= f(x)g(x)\Big|_{x=0}^{x=1} - \int_0^1 f(x)g'(x)\,dx$$

$$= f(1)g(1) - f(0)g(0) + \int_0^1 f(x)[-g'(x)]\,dx$$

$$= \underbrace{f(1)g(1) - f(0)g(0)}_{\text{can't be expressed in terms of an } L^2\text{-inner product}} + \langle f, -\tfrac{dg}{dx}\rangle$$

If we restricted $D(L)$ so that $f(1)g(1) - f(0)g(0) = 0$, then we could say that $\langle f, -\frac{dg}{dx}\rangle = \langle f, L^* g\rangle$ with $L^* g = -\frac{dg}{dx}$. But with $D(L) = H^1[0,1]$, L^* is not well-defined.

__Defn.3.3__ A subspace $M \subset H$ is a __closed__ __linear__ __manifold__ if every sequence in M which is convergent in H is also convergent in M (So that M is a Hilbert space itself)

__Ex 1:__ Let L be a bdd. linear operator, then $N(L)$ is a closed linear manifold. (See your textbook, pg. 107)

Ex2: Spse M is a Linear manifold in H (i.e. a subspace of H)
Define the <u>ORthogonal</u> <u>complement</u> <u>of M</u>
to be
$$M' = \{ f \in H \mid \langle f, g \rangle = 0 \text{ for all } g \in M \}$$

(<u>Note</u>: this is the analog of our $M^{\perp}$ for finite dimensional vector spaces!!)

<u>Claim</u>: M' is a closed Linear manifold.

Pf: Spse $\{u_n\}$ is a sequence in M', and spse that $\lim_{n \to \infty} u_n = u \in H$.

(We want to show that $u \in M'$ also, i.e. $\langle u, g \rangle = 0 \; \forall g \in M$)
Let $g \in H$, and note that $L_g : H \to \mathbb{R}$ defines a bdd. Linear operator
$$L_g u = \langle u, g \rangle \quad \forall u \in H$$
(See Ex 4 of Bdd Lin. Oper. Notes)

Since $u_n, u \in H$, we examine
$$L_g(u_n - u) = \langle u_n - u, g \rangle$$
Note that L_g is a bdd. operator so that
$$|L_g(u_n - u)| = |\langle u_n - u, g \rangle| \le \|u_n - u\| \, \|g\|$$
Since $g \in H$, $\|g\| < \infty$. Given $\varepsilon > 0$, we can choose N so that for $n > N$, we have $\|u_n - u\| < \varepsilon / \|g\|$.
Hence, for $n > N$, we have
$$|\langle u_n - u, g \rangle| \le \|u_n - u\| \, \|g\| < \frac{\varepsilon}{\|g\|} \cdot \|g\| = \varepsilon$$
$$\Rightarrow \lim_{n \to \infty} \langle u_n - u, g \rangle = 0.$$

That is, $\lim_{n\to\infty} \langle u_n, g\rangle - \langle u, g\rangle = 0$

$$\underbrace{\lim_{n\to\infty} \langle u_n, g\rangle}_{=0\ \forall u_n \text{ since } u_n \in M'} = \langle u, g\rangle.$$

Hence, $\lim_{n\to\infty} \langle u_n, g\rangle = 0$, and we've shown that

$$\langle u, g\rangle = 0.$$

Since, g was an arbitrary element of M, we've shown that $\langle u, g\rangle = 0\ \forall g \in M$ so that $u \in M'$, and thus, M' is a closed linear manifold. So M' is a Hilbert space. //

Just as we had direct sums $S \oplus S^\perp$ in finit dimensional vector spaces, we have a similar result for Hilbert spaces.

Let M denote a linear manifold of H. Then if $f \in H$ $\exists$ $f_1 \in$ closure of M and $f_2 \in M'$ so that

$$f = f_1 + f_2$$

And this decomposition is unique.

Note: You've already seen a few examples of linear functionals, but in the next defn, Keener makes it official →

Defn 3.4 (1.) A Linear Functional $T: H \to \mathbb{R}$ is a linear mapping of H into $\mathbb{R}$ satisfying

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg \quad \forall \alpha, \beta \in \mathbb{R} \text{ and } \forall f, g \in H$$

(2.) A Linear Functional $T: H \to \mathbb{R}$ is Bounded if there is a positive real # C for which:

$$|Tu| \leq C\|u\| \quad \forall\, u \in H.$$

Recall Ex. 4 from the Bdd. Lin. Operator Notes:
Fix $g \in L^2[0,1]$, then $T_g : L^2[0,1] \to \mathbb{R}$ defined by

$$T_g(f) = \langle g, f \rangle = \int_0^1 g(x) f(x)\, dx$$

is a Bdd. Linear Functional on $L^2[0,1]$

Note: A Bdd. Linear Functional is a special Case of a Bdd. Linear Operator where the Range of L happens to be $\mathbb{R}$. (or a subset of $\mathbb{R}$)

The example above shows that any $g \in L^2[0,1]$ determines a bdd. linear functional on $L^2[0,1]$. The following theorem says the converse... ie. that all bdd, linear fctnals on a Hilbert space have a "Representation" in terms of this inner product $\langle g, f \rangle$.

* The Riesz Representation Theorem is one of those fundamental theorems of Applied Math

(H is a Hilbert Space)

Thm 3.1 <u>Riesz Representation Theorem</u>

For any bounded linear functional $T: H \to \mathbb{R}$, there exists a unique $g \in H$ so that $Tf = \langle f, g \rangle \ \forall f \in H$.

<u>Remark</u>: Riesz Rep. Thm says that if we are given a bdd linear fctnal T that maps the elements of H into $\mathbb{R}$, then we can always find a ! element of H, call it g, so that "The action of the linear functional, T, on the elements of H can be <u>Represented</u> by the inner-product of g with the elements of H." i.e.

$$Tf = \langle f, g \rangle \ \forall f \in H$$

the action of T on some $f \in H$ & it can be represented as an inner-product of f with g

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<u>pf</u>: (<u>Uniqueness First</u>) Spse $\exists \ g_1, g_2 \in H$ so that

$$Tf = \langle f, g_1 \rangle \ \forall f \in H \text{ and } Tf = \langle f, g_2 \rangle \ \forall f \in H.$$

So,
$$Tf = \langle f, g_1 \rangle = \langle f, g_2 \rangle \ \forall f \in H$$
$$\Rightarrow \langle f, g_1 - g_2 \rangle = 0 \ \forall f \in H \text{ by properties of } \langle \cdot, \cdot \rangle$$

But $g_1 - g_2 \in H$, so this implies that

$$\langle g_1 - g_2, g_1 - g_2 \rangle = 0 \Rightarrow \|g_1 - g_2\|^2 = 0 \text{ by defn of } \|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$$
$$\Rightarrow \|g_1 - g_2\| = 0$$

So, $g_1 - g_2 = 0 \in H \Rightarrow g_1 = g_2 \in H$, and the $g \in H$ is unique!

(Now we show the existence of $g \in H$)

Recall the null space of T as

$$N = \{f \in H \mid Tf = 0\}$$

$\hookrightarrow 0 \in \mathbb{R}$ since $T: H \to \mathbb{R}$

Case 1: If $N = H$, then $Tf = 0 \; \forall f \in H$, and we can choose $g = 0 \in H$ to obtain

$$\underset{0 \in \mathbb{R}}{0} = \langle f, \underset{0 \in H}{0} \rangle = \langle f, g \rangle = Tf \quad \forall f \in H$$

and we are done since $Tf = \langle f, g \rangle \; \forall f \in H$

Case 2: If N is a proper subset of H, then N' (orthog. comp. of N) is a nontrivial, closed subspace of H. (We showed this on page 4 of these notes.)

Hence, we choose $g_0 \in N'$ with $\|g_0\| = 1$.

Then $\langle g_0, f \rangle = 0 \quad \forall f \in N$ by defn. of N.

Note $g_0 \neq 0$ since $g_0 \in N'$ with $\|g_0\| = 1$, and

$Tg_0 \neq 0$ since $g_0 \notin N$

Define $g = (Tg_0) g_0$ (Note $Tg_0 \in \mathbb{R}$ is a nonzero scalar)

(Now, we want to show that $Tf = \langle f, g \rangle \; \forall f \in H$.)

So, for any $f \in H$, we let

$$y = (Tf) g_0 - (Tg_0) f \in H \text{ since } g_0, f \in H, \; Tf, Tg_0 \in \mathbb{R}.$$

Claim: $y \in N$

pf:
$$\begin{aligned} Ty &= T[(Tf) g_0 - (Tg_0) f] \\ &= (Tf) Tg_0 - (Tg_0) Tf \qquad \text{But } Tf, Tg_0 \in \mathbb{R} \text{ so that} \\ &= (Tf) Tg_0 - (Tf) Tg_0 \\ &= 0 \end{aligned}$$

//

But if $y = (Tf)g_0 - (Tg_0)f \in N$, then $\langle y, g_0 \rangle = 0$ since $g_0 \in N'$

So,

$$\begin{aligned} 0 &= \langle y, g_0 \rangle \\ &= \langle (Tf)g_0 - (Tg_0)f, g_0 \rangle \\ &= \langle (Tf)g_0, g_0 \rangle - \langle (Tg_0)f, g_0 \rangle \\ 0 &= (Tf)\langle g_0, g_0 \rangle - (Tg_0)\langle f, g_0 \rangle \end{aligned}$$

$\Rightarrow$ $(Tf)\langle g_0, g_0 \rangle = (Tg_0)\langle f, g_0 \rangle$ since $Tg_0 \in \mathbb{R}$

$\Rightarrow$ $(Tf)\underbrace{\|g_0\|^2}_{=1 \text{ by construction}} = \langle f, (Tg_0) g_0 \rangle$

So, $Tf = \langle f, (Tg_0)g_0 \rangle$

$= \langle f, g \rangle$ by defn of g

Hence, $Tf = \langle f, g \rangle \quad \forall f \in H$ //

Remark: Riesz Representation Theorem fails to apply for an Unbounded Linear functional.