Structure of the global attractor of cyclic feedback systems

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Abstract

We characterize the dynamics on global attractors of cyclic feedback systems. Under mild restrictions the description is given in terms of a semiconjugacy to a simple model system which posseses Morse-Smale dynamics. However, for the completely general case, no simple model system is feasible and hence we introduce a weaker notion of equivalence, namely, topological semi-equivalency. We then prove that the global attractor of a cyclic feedback system is topologicaly semi-equivalent to the original model flow.

Main ingredients in the proof are the discrete Lyapunov functional introduced by Mallet-Paret and Smith, and the Conley index theory.

Key words: Cyclic feedback system, Conley index theory, Lyapunov functional, global dynamics.

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†Research was supported in part by NSF Grant DMS-9101412.; Center for Dynamical Systems and Nonlinear Studies , School of Mathematics , Georgia Institute of Technology , Atlanta, GA 30332

1 Introduction

The goal of this paper is to describe the global dynamics of *cyclic feedback systems* ($\mathcal{CFS}$), i.e. systems of ordinary differential equations of the form

$$\dot{x_i} = f_i(x_i, x_{i-1}) \quad i = 1, \ldots, n \quad (x_0 = x_n) \tag{1}$$

where

$$f_i \in C^1(\mathbf{R}^n, \mathbf{R}) \quad i = 1, \ldots, n; \tag{2}$$

for all $v \neq 0$,

$$\begin{aligned} \delta_1 f_1(0, v)v &> 0 \quad \delta_1 \in \{+1, -1\} \\ f_i(0, v)v &> 0 \quad i = 2, \ldots, n; \end{aligned} \tag{3}$$

and

$$\begin{aligned} \delta_1 \frac{\partial f_1(\eta, \zeta)}{\partial \zeta}|_{(0,0)} &> 0 \\ \frac{\partial f_i(\eta, \zeta)}{\partial \zeta}|_{(0,0)} &> 0 \quad i = 2, \ldots, n. \end{aligned} \tag{4}$$

For $i = 2, \ldots, n$, let $\delta_i = 1$. As will be seen the sign of δ_1 plays an important role in the global dynamics of these systems. Thus, these systems will be denoted by $\mathcal{CSF}^{\pm}$ when $\delta_1 = \pm 1$, respectively.

A more general form of cyclic feedback systems is (1) in conjuction with the assumptions that for $v \neq 0$, $\delta_i f_i(0, v)v > 0$ and $\delta_i \frac{\partial f_i(\eta,\zeta)}{\partial \zeta}|_{(0,0)}$ where $\delta_i \in \{\pm 1\}$. The reader is referred to Mallet-Paret and Smith (1990) and Hofbauer et. al. (1991), and the references therein for a sampling of the types of phenomena modeled by this class of equations. These systems are characterized by $\Delta = \delta_1 \delta_2 \ldots \delta_n$. However, since an appropriate change of variables of the form $x_i \to \lambda_i x_i$, $\lambda_i \in \{+1, -1\}$ results in (3) where $\delta_1 = \Delta$ there is no loss of generality in working with $\mathcal{CFS}$.

A special, but important subclass of $\mathcal{CFS}$, are *monotone cyclic feedback systems* ($\mathcal{MCFS}$) where the local assumption (4) is replaced by

$$\begin{aligned} \delta_1 \frac{\partial f_1(\eta, \zeta)}{\partial \zeta} &> 0 \\ \frac{\partial f_i(\eta, \zeta)}{\partial \zeta} &> 0 \quad i = 2, \ldots, n. \end{aligned} \tag{5}$$

There are two tools which make our analysis possible:

1. an integer valued Lyapunov function, and
2. the Conley index theory.

Though each will be discussed in greater detail later in the paper, we take this opportunity to outline the role that each plays in our analysis.

We begin by discussing the Lyapunov function. Let $\sigma_i \in \{\pm 1\}$ and set

$$Q(\sigma_1, \ldots, \sigma_n) = \{x \in \mathbf{R}^n \mid \sigma_i x_i > 0\}.$$

Observe that $Q(\sigma_1, \ldots, \sigma_n)$ is an open cone corresponding to an "orthant" in $\mathbf{R}^n$. Define $\mathcal{N} : \bigcup Q(\sigma_1, \ldots, \sigma_n) \to \mathbf{Z}$ by

$$\mathcal{N}(x) = card\{i \mid \delta_i x_i x_{i-1} < 0\}. \tag{6}$$

Let

$$X_i = \{x \in R^n \mid x_i = 0, \delta_{i+1}\delta_i x_{i+1} x_{i-1} < 0\}.$$

It is easy to see, that one can extend the domain of definition of $\mathcal{N}$ (by continuity) to

$$X := (\cup X_i)) \cup (\cup Q(\sigma_1, \ldots, \sigma_n)).$$

A geometrical view of $\mathcal{N}$ may be enlightening. $\mathcal{N}$ is constant on every orthant $Q(\sigma_1, \ldots, \sigma_n)$ and the hyperplanes X_i represent the boundaries of the orthants where $\mathcal{N}$ agrees. Thus, on each component of X, $\mathcal{N}$ is constant and the value of $\mathcal{N}$ differs on any two components. Furthermore, $\mathbf{R}^n \setminus X$ consists of the boundary points between components on which $\mathcal{N}$ differs. Thus, on the complement of X, $\mathcal{N}$ is left undefined. Observe, also, that for those $x \in R^n$ with each $x_i \neq 0$, $1 \leq i \leq n$

$$(-1)^{\mathcal{N}(x)} = sign \prod_{i=1}^{n} \delta_i x_i x_{i-1} = \prod_{i=1}^{n} \delta_i = \delta_1 \tag{7}$$

so $\mathcal{N}$ takes only odd values if $\delta_1 = -1$ and only even values if $\delta = 1$.

The following result justifies the name Lyapunov functional for N.

Proposition 1.1 [Mallet-Paret and Smith (1990)] *Let $x(t)$ be a nontrivial solution of (1). Then*

a. $x(t) \in X$ except at isolated values of t.

b. $\mathcal{N}(x(t))$ is locally constant for $x(t) \in X$.

c. if $x(t_0) \notin X$ then $\mathcal{N}(x(t_0^+)) < \mathcal{N}(x(t_0^-))$ where $t_0^+ > t_0$ and $t_0^- < t_0$.

d. if $x(t) \in X$ then $(x_i(t), x_{i-1}(t)) \neq (0,0)$ for $1 \leq i \leq n$.

The Lyapunov function of this form was used by Mallet-Paret and Smith (1990) to show that in the case of $\mathcal{MCFS}$ the only possible invariant sets are fixed points and periodic orbits. For $\mathcal{MCFS}$ with $\Delta = +1$ Fusco and Oliva (1990) showed, that for any two hyperbolic periodic orbits o^- and $o+$ unstable manifold $W^u(o^-)$ intersect the stable manifold $W^s(o^+)$ transversaly. Their theorem can be adapted to the case $\Delta = -1$.

Turning now to the ideas of the index theory, C. Conley (1978) recognized that given a compact invariant set $\mathcal{A}$ there exists a natural decomposition into subinvariant sets defined as follows.

Definition 1.2 [C. Conley (1978)] Let S be a compact metric space with a flow denoted by $x \cdot t$, where $x \in S$ and $t \in (-\infty, \infty)$, and let $\alpha(x)$ and $\omega(x)$ denote the alpha and omega limit sets of the orbit through x. A *Morse decomposition* of S is a finite ordered collection

$$\mathcal{M}(S) = \{M(p) \mid p = 0, \ldots, P\}$$

of disjoint compact invariant subsets of S such that

$$x \in S \Rightarrow \text{ there exists } p \geq q \text{ such that } \alpha(x) \subset M(p) \text{ and } \omega(x) \subset M(q)$$

and furthermore

$$p = q \Rightarrow x \in M(p), \text{ hence } x \cdot t \in M(p) \text{ for all } t.$$

The individual invariant subsets $M(p)$ are called *Morse sets*, and the remaining portion, $S \setminus \bigcup M(p)$, is referred to as the set of *connecting orbits*.

We shall make the following assumption (which can be viewed as a condition on the nonlinearity at infinity) throughout this paper.

C1 *There exists a global compact attractor $\mathcal{A}$ for $\mathcal{CFS}$.*

It is the dynamics on this attractor $\mathcal{A}$ which we intend to describe.

Combining the ideas of Conley with the Lyapunov function, one is tempted in the case $\Delta = -1$ to define a Morse decomposition for $\mathcal{A}$ by setting

$$M(p) := \{x \in \mathbf{R}^n \mid \mathcal{N}(\varphi(t,x)) = 2p+1 \; \forall\, t \in \mathbf{R}, \; p = 0, \ldots\}.$$

This almost works. Observe that (3) forces the origin to be a fixed point for (1), and hence, it must lie in a Morse set. However, $\mathcal{N}(\mathbf{0})$ is not defined. This problem is resolved by setting

$$M(P) = \{\mathbf{0}\} \bigcup \left(\bigcup_{p \geq P} M(p) \right)$$

where P is related to the dimension of the unstable manifold at $\mathbf{0}$. For the purposes of this introduction

$$\mathcal{M}(\mathcal{A}) := \{M(p) \mid 0 \leq p \leq P\}$$

will be taken as the Morse decomposition of $\mathcal{A}$.

Our global description of the dynamics of the cyclic feedback system will be given in terms of the Morse decompositions. Thus the first question which should be addressed is

Q1. *What type of dynamics occurs in the Morse sets?*

The importance of using Morse decompostions as a framework becomes evident at this point. There is no "universal" answer to **Q1**. In the case of $\mathcal{MCFS}$ the work of Mallet–Paret and Smith (1990) shows that a Morse set contains at most fixed points, periodic orbits, and connecting orbits between them. On the other hand the work of Gedeon (1994) shows that it is possible to have Morse sets which exhibit chaotic dynamics for relatively simple $\mathcal{CFS}$. The concept of Morse set has enough flexibility to allow for this large variety of dynamics.

To state our results concerning **Q1** we need to introduce some notation. Let

$$\varphi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$$

denote the flow generated by the $\mathcal{CFS}$. Given $N \subset \mathbf{R}^n$ its *maximal invariant set* is

$$\mathrm{Inv}N := \{x \in \mathbf{R}^n \mid \varphi(\mathbf{R}, x) \subset N\}.$$

If N is a compact set such that $\mathrm{Inv}N \subset \mathrm{int}N$, the N is called an *isolating neighborhood* and $\mathrm{Inv}N$ is an *isolated invariant set*. Since a Morse set is an isolated invariant set, given $M(p)$ there exists N_p such that

$$M(p) = \mathrm{Inv}N_p \subset \mathrm{int}N_p.$$

Recall that $\Xi \subset \mathbf{R}^n$ is a *local section* for φ if there exists a time $T > 0$ such that the set $\varphi([-T, T], \Xi)$ is homeomorphic to $[-T, T] \times \Xi$. Ξ is a *Poincaré section* for a Morse set $M(p)$ if it is a local section; given an isolating neighborhood N_p of $M(p)$, $\Xi \cap N_p$ is closed; and for every $x \in M(p)$, $\varphi((0, \infty), x) \cap \Xi \neq \emptyset$.

For $\mathcal{CFS}$ there are ideal candidates for Poincaré sections. Observe that on X_i, $\dot{x}_i \neq 0$. Thus compact subsets of X_i's are local sections for the flow. It is not known in general under what conditions X_i acts as a Poincaré section for a Morse set. However, if it does act as a Poincaré section, then a periodic orbit $x(t)$ in $M(p)$ can be characterized as follows. For every $i = 1, \ldots, n$ there exist times t_i and t_i' such that $x_i(t_i) > 0$ and $x_i(t_i') < 0$. We shall refer to periodic orbits with this property as *large periodic orbits*. If X_i is not a Poincaré section for $M(p)$, then it appears possible that there exist periodic orbits which remain in an orthant $Q(\sigma_1, \ldots, \sigma_n)$. We shall refer to such a periodic orbit as a *small periodic orbit*.

Theorem 1.3 [McCord et.al, Corollary 1.4] *If M is an isolated invariant set with the Conley index of a hyperbolic periodic orbit and if M has a Poincaré section, then M contains a periodic orbit.*

This theorem is used to prove, at least in part, the following theorem.

Theorem 1.4 *For $\mathcal{CFS}^+$ let $p = 1, \ldots, P-1$ and for $\mathcal{CFS}^-$ let $p = 0, \ldots, P-1$*

1. *There exists a continuous surjective map*

   $$\theta_p : M(p) \to S^1$$

   where S^1 is the unit circle.

2. *If, for some $i = 1, \ldots, n$, the set X_i is a Poincaré section of $M(p)$, then $M(p)$ contains a large periodic orbit.*

3. *If in addition one considers a $\mathcal{MCFS}$ and if $M(p)$ contains no fixed points then $M(p)$ contains a large periodic orbit.*

Proof of parts 2 and 3: As will be shown in Lemma 4.7, $M(p)$ has the index of a hyperbolic periodic orbit. In 2 it is assumed that $M(p)$ has a Poincaré section. Hence by Theorem 1.3 $M(p)$ contains a periodic orbit and by Corollary 5.3 this periodic orbit is large. In the case of $\mathcal{MCFS}$ Lemma 5.1 states that $M(p)$ has a Poincaré section. Again, the result follows from Theorem 1.3 and Corollary 5.3. □

The proof of the first part of Theorem 1.4 requires more work and can be found in Section 6.

We remark that we have not yet constructed an example of a $\mathcal{CFS}$ with a Morse set which contains no fixed points and yet contains a small periodic orbit.

With this admittedly weak understanding of the Morse sets the next question is:

Q2 *What is the structure of the set of connecting orbits?*

The answer to this question will be given in two forms. The first takes the form of a semi–conjugacy of the flow on $\mathcal{A}$ onto a well understood flow on a unit disk of an appropriate dimension which we shall refer to as the model flow.

This model flow is defined as follows. Let A be an $k \times k$ matrix of the form

$$A = \begin{bmatrix} A_0 & 0 & \dots & 0 \\ 0 & A_1 & & \\ & & \ddots & \\ 0 & & & A_{P-1} \end{bmatrix}.$$

The submatrices A_p, $p = 0, \dots, P-1$ have two forms:

$$A_p = \frac{1}{p+1} \qquad \text{(Type I)}$$

and

$$A_p = \begin{bmatrix} (p+1)^{-1} & 2\pi \\ -2\pi & (p+1)^{-1} \end{bmatrix}. \qquad \text{(Type II)}$$

Let $z = (z_0, \dots, z_{k-1}) \in \mathbf{R}^k$. Then in polar coordinates $z = r\zeta$ where $r \geq 0$ and $\zeta \in S^{k-1}$, the unit sphere in $\mathbf{R}^k$. Let $D^k = \{z = (z_0, \dots, z_{k-1}) \mid \sum_{p=0}^{k-1} z_p^2 \leq 1\}$ be the closed unit ball in $\mathbf{R}^k$. Consider the flow

$$\psi : \mathbf{R} \times D^k \to D^k \tag{8}$$

generated by the equations

$$\dot{\zeta} = A\zeta - \langle A\zeta, \zeta \rangle \zeta \tag{9}$$

$$\dot{r} = r(1-r). \tag{10}$$

The dynamics of ψ is most easily understood if one observes that (9) is obtained by projecting the linear system $\dot{z} = Az$ onto the unit sphere.

The value of k and the choice of Type I or Type II matrices is determined by the $\mathcal{CFS}$. In particular k is related to the number of eigenvalues with positive real part of the matrix obtained by linearizing about $\mathbf{0}$ in the $\mathcal{CFS}$. We shall denote the cyclic feedback system by $\mathcal{CFS}_{even}$ and $\mathcal{CFS}_{odd}$ if n is even or odd, respectively. The specific choices for the A_p's as a function of the type of $\mathcal{CFS}$ are as follows:

$\mathcal{CFS}^-_{odd}$: A_p, $p = 0, \dots P-1$ are of Type II unless $n = 2P+1$ when A_p, $p = 0, \dots, P-2$ are of Type II and A_{P-1} is of Type I.

$\mathcal{CFS}^+_{odd}$: A_0 is of Type I and A_p, $p = 1, \dots P-1$ are of Type II.

$\mathcal{CFS}^-_{even}$: A_p, $p = 0, \dots P-1$ are of Type II.

$\mathcal{CFS}^+_{even}$: A_0 is of Type I and A_p, $p = 1, \dots, P-1$ are of Type II unless $n = 2P$ when A_p, $p = 1, \dots, P-2$ are of the Type II and A_{P-1} is of Type I.

When it is necessary to distinguish between the model flows we shall let $\psi^\pm_*$ denote the corresponding flow where $*$ denotes *even* or *odd*.

Let $\Pi(p)$, $p = 0, \dots, P-1$ denote the invariant set of ψ in the invariant subspace corresponding to A_p and let $\Pi(P) := \mathbf{0}$, the origin. Observe that $\{\Pi(p) \mid p = 0, \dots, P\}$ forms a Morse decomposition of ψ on D^k.

Theorem 1.5 *Consider $\mathcal{CFS}^\pm_*$. Assume that if A_p is of Type II, then $M(p)$ has a Poincaré section. Then, there exist a continuous surjective function*

$$\rho : \mathcal{A} \to D^K$$

for which $M_p = \rho^{-1}(\Pi(p))$ $(p = 0, \dots, P)$ and a continuous flow $\tilde{\varphi} : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$ obtained via an order preserving time reparameterization of φ such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{R} \times \mathcal{A} & \xrightarrow{id \times \rho} & \mathbf{R} \times D^K \\ \downarrow \tilde{\varphi} & & \downarrow \psi^\pm_* \\ \mathcal{A} & \xrightarrow{\rho} & D^K \end{array}$$

i.e. φ is semi–conjugate to $\psi^\pm_*$.

An immediate corollary of Theorem 1.4 and Theorem 1.5 is as follows.

Corollary 1.6 *Consider $\mathcal{MCFS}^\pm_*$ and assume that if A_p is of Type II, then $M(p)$ has no fixed points. Then, there exists a semi–conjugacy from $\tilde{\varphi}$ to $\psi^\pm_*$.*

We want to take some time to explain the implications of Theorem 1.5 to the dynamics of $\mathcal{CFS}$.

One conclusion follows immediately from Theorem 1.4.2 ; if $M(p)$ is mapped by the semi–conjugacy ρ to a periodic orbit, then $M(p)$ contains a large periodic orbit.

However, a set $M(p)$ may be more complicated then a periodic orbit. In fact Gedeon (1994) constructed the following example of a $\mathcal{CFS}$ where the set $M(0)$ is at least a suspension of a shift dynamics on two symbols.

Consider a $\mathcal{CFS}$ system of the form

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - b_1 f(x_3) \\ \dot{x}_2 &= -a_2 x_2 + b_2 x_1 \\ \dot{x}_3 &= -a_3 x_3 + b_3 x_2 \end{aligned}$$

with $f'(0) > 0$, $xf(x) > 0$ and where f is a C^1 function of the form

$$f(x) = \begin{cases} \text{ is increasing for } x \in (-\infty, A] \\ 0 < f(x) \leq \text{L for } x \in (A+\eta, \infty) \text{ and decreasing} \\ 0 < f(x) \text{ and decreasing for } x \in (A, A+\eta] \end{cases} \qquad (11)$$

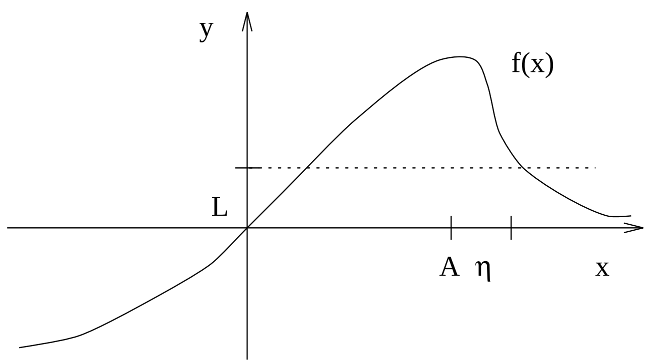

Figure 1: A function $f(x)$.

Theorem 1.7 [Gedeon (1994)] *There is a function f of the form (11) and a Poincaré section H with a Poincaré map π with the following property. There is an invariant set $S_f \subset H$ under a Poincaré map π, a surjective map $h : S_f \to \Sigma_2$ and a positive integer d such that following diagram comutes*

$$\begin{array}{ccc} \mathbf{S}_f & \xrightarrow{\pi^d} & \mathbf{S}_f \\ h \downarrow & & \downarrow h \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

where (Σ_2, σ) represents the full shift dynamics on two symbols.

Let us remark, that the complicated behaviour is confined to the Morse set $M(0)$. As we see in this case the semi–conjugacy ρ maps a complicated invariant set $M(0)$ onto a periodic orbit in the model flow. See Figure 2 below.

In this particular example we lost a tremendeous amount of information via the semi–conjugacy. On the other hand, this deliberate negligence allowed us to characterize the dynamics on the global attractor for the entire class of $\mathcal{CFS}$.

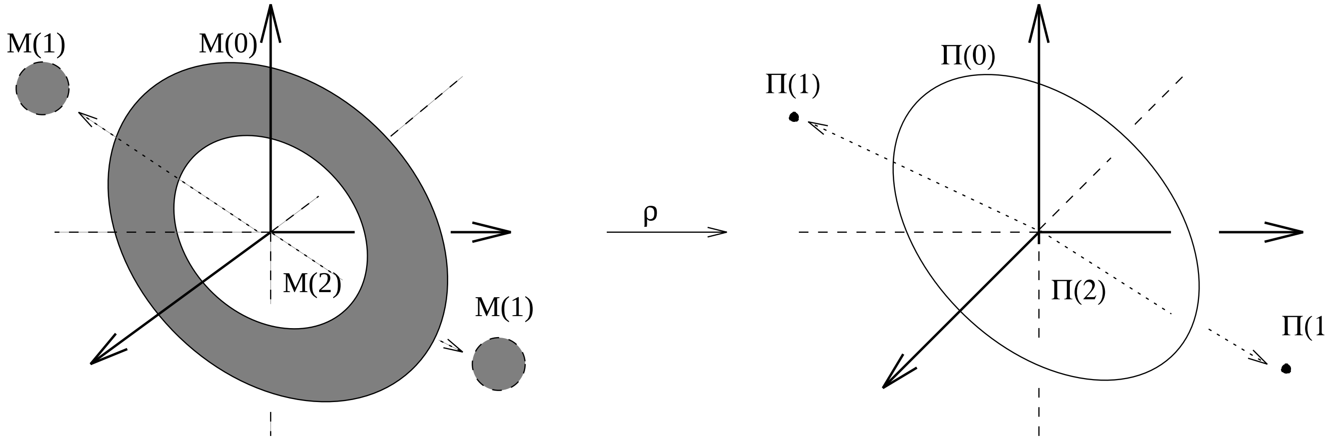

Figure 2: Semi–conjugacy in the case $n = 3$.

The other implication of Theorem 1.5 and Corollary 1.6 concerns the existence of connecting orbits between Morse sets. For the model flow ψ the stable and unstable manifolds for the Morse sets $\Pi(p)$ are known explicitly. Hence the set of connecting orbits between any two Morse sets, $C(\Pi(p), \Pi(q))$, is also known explicitly.

Returning to the cyclic feedback system we now conclude that

$$C(M(p), M(q)) = \rho^{-1} C(\Pi(p), \Pi(q)).$$

In analogy with the discussion above, how well $C(\Pi(p), \Pi(q))$ "approximates" $C(M(p), M(q))$ depends on how much dynamics is being lost through the semi–conjugacy. In particular, if we retun to the example generated by (11) then the set of connecting orbits $C(M(12), M(0))$, (where $M(12) := M(2) \cup M(1) \cup C(M(2), M(1)))$, through the preimage of $T^2 \times (0, 1)$ (T^2 is the two torus), must be fairly complicated since the suspension of S_f is contained in the closure of $C(M(12), M(0))$.

Obviously, the hypotheses of Theorem 1.5 will not always be satisfied; there may occur fixed points in $M(p)$ and even if there are no fixed points we conjecture that there are $\mathcal{CFS}$ for which $M(p)$ will not have a Poincaré section. In these cases we have no hope of controlling the dynamics in the Morse sets and hence there is no hope of constructing a semi–conjugacy. There are two ways how to address this problem. We can restrict the class of systems and show that in the restricted class the hyperplanes X_i act as a Poincarè sections for appropriate Morse sets. We can also introduce a weaker notion of comparison between invariant sets of flows.

Let us follow both ideas.

Theorem 1.8 *(Sufficient condition for the existence of Poincarè sections.)*
Consider $\mathcal{CFS}$ of the form

$$\dot{x}_i = \alpha_i g_i(x_i) + \beta_i f_i(x_{i-1}), \qquad i = 1, \dots, n$$

$\alpha_i, \beta_i \in \{\pm 1\}$ *and we assume that for every i $x_i g_i(x_i) > 0$ and $x_{i-1} f_{i-1} > 0$. Let $u_i = -\alpha_i \beta_i$.*
If $\prod_{i=1}^n u_i = -1$ then for every i X_i is a Poincarè section.

The proof of Theorem 1.8 can be found at the end of section 5.

To follow the second idea, we first recall that given a Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ an *interval* $I \subset \mathcal{P}$ satisfies the property that if $p, q \in I$ and $p > r > q$, then $r \in I$. The importance of intervals is that given a Morse decomposition all coarser Morse decompositions involve isolated invariant sets of the form

$$M(I) := \left(\bigcup_{p \in I} M(p) \right) \bigcup \left(\bigcup_{p,q \in I} C(p,q) \right)$$

where I is an interval and

$$C(p,q) := \{x \in \mathcal{A} \mid \omega(x) \subset M(q) \text{ and } \alpha(x) \subset M(p)\}$$

is the set of *connecting orbits* from $M(p)$ to $M(q)$.

Definition 1.9 If $\mathcal{A}$ and $\mathcal{B}$ are invariant sets with Morse decompositions $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ and $\mathcal{M}(\mathcal{B}) = \{M(q) \mid q \in (\mathcal{Q}, >)\}$ respectively, then the Morse decomposition $\mathcal{M}(\mathcal{A})$ is *topologically semi-equivalent* to the Morse decomposition $\mathcal{M}(\mathcal{B})$ if there exists

1. an order preserving bijection $\bar{\rho} : \mathcal{P} \to \mathcal{Q}$, and
2. a continuous surjection $\rho : \mathcal{A} \to \mathcal{B}$ such that

   $$M(I) = \rho^{-1}(M(\bar{\rho}(I)))$$

   for every interval $I \subset \mathcal{P}$.

Theorem 1.10 *Given $\mathcal{CFS}_*^{\pm}$, the Morse decomposition $\mathcal{M}(\mathcal{A})$ is topologically semi-equivalent to $\mathcal{M}(D^K, \psi_*^{\pm})$.*

Acknowledgment: We would like to thank Chris McCord for helpful discusions related to Section 6.

2 Linear theory

The discrete Ljapunov function $\mathcal{N}$ described in the introduction has its roots in the linear theory of $\mathcal{CFS}$ which will be described in this section. A corresponding theory for discrete linear systems can be found in Mallet-Paret and Smith (1990).

Consider the system

$$\dot{x} = Ax \tag{12}$$

and assume that it is a linear cyclic feedback system i.e.

$$a_{ij} = 0 \quad \text{for } j \neq i, i-1.$$

In this setting conditions (3) and (4) take the form

$$\delta_i a_{i,i-1} > 0 \text{ for } i = 1, \ldots, n \pmod{n}.$$

Let us define complex eigenspaces of A by

$$\Phi_\sigma = Ker(A - \sigma I) \subset C^n$$

$$\Psi_\sigma = gen\ ker(A - \sigma I) \subset C^n$$

where $gen\ ker\ B = ker\ B^m$ for sufficiently large m is the generalized kernel of the matrix B. For fixed α, define

$$F_\alpha = Re \bigoplus_{Re\,\sigma=\alpha} \Phi_\sigma$$

and

$$P_\alpha = Re \bigoplus_{Re\,\sigma=\alpha} \Psi_\sigma,$$

the real parts of the spans.

Lemma 2.1 *Given α there exists an integer k such that for each initial condition $x_0 \in F_\alpha \setminus \{0\}$ the solution $x(t)$ of (12) satisfies $N(x(t)) = k$ for all $t \in \mathbf{R}$. Furthermore, all zeroes of $x_i(t)$ are simple for each i.*

Proof. Observe that $\mathcal{N}(x(t)) = \mathcal{N}(cx(t))$ for any constant $c > 0$.

If $x_0 \in F_\sigma \setminus \{0\}$ then the solution can be written as $x(t) = e^{\alpha t}(x_0 + \tilde{q}(t))$ where the first term comes from the eigenvalues with zero imaginary part and second with nonzero imaginary part. One can see that $q(t) := x_0 + \tilde{q}(t)$ is, in general, quasiperiodic.

Fix $t_0 \in R$ such that $q(t_0) \in X$. Then, there exists $t_1 < t_0 < t_2$ with $|t_1|, |t_2|$ arbitrarily large such that $q(t_1)$ and $q(t_2)$ are in the same component of X as $q(t_0)$.

Since $\mathcal{N}$ is constant on each component of X we have

$$\mathcal{N}(x_{t_j}) = \mathcal{N}(e^{\alpha t_j} q(t_j)) = \mathcal{N}(q(t_j)) = \mathcal{N}(q(t_0)), \ \ j = 1, 2$$

hence

$$\mathcal{N}(x(t_1)) = \mathcal{N}(x(t_2)).$$

The monotonicity of $\mathcal{N}(x(t))$ in t implies that $\mathcal{N}(x(t))$ is a constant $k(x_0)$ independent of t.

Since $k(x_0)$ is a well defined integer for $x_0 \in F_\alpha \setminus \{0\}$ and it is locally constant, $k(x_0) = k$ independent of x_0.

Simplicity of the zeroes follows from Proposition 1.1. □

Lemma 2.2 *The statement of Lemma 2.1 holds, with P_α replacing F_α.*

Proof. If $x_0 \in P_\alpha \setminus \{0\}$ then $x(t) = e^{\alpha t}(p(t) + u(t))$ where $p(t)$ is a polynomial in t and $u(t)$ has terms of the form $a(t)T(t)$ where $a(t)$ is a polynomial and $T(t)$ is a trigonometric function. Again, $p(t)$ comes from the Jordan blocks coresponding to the eigenvalue α and $u(t)$ comes from the blocks with eigenvalues with real part α and nonzero imaginary parts. Let a be the highest degree of polynomial expression $p(t) + u(t)$. Then

$$x(t) = t^a e^{\alpha t} q(t) + O(t^{a-1} e^{\alpha t})$$

as $t \to \pm\infty$ where $q(t)$ is quasiperiodic.

Indeed, if *max deg* $p(t) > $ *max deg* $u(t)$ then $q(t)$ is a constant and if *max deg* $p(t) \leq$ *max deg* $u(t)$ then $q(t)$ is quasiperiodic.

With t_1 and t_2 chosen as above and sufficiently large, one has for $j = 1, 2$ that $t_j^{-a} e^{-\alpha t_j} x(t_j)$ is arbitrary close to $q(t_j)$, and hence to $q(t_0)$. Thus

$$\mathcal{N}(x(t_j)) = \mathcal{N}(t_j^{-a} e^{-\alpha t_j} x(t_j)) = \mathcal{N}(q(t_0)) = k.$$

Thus the local constancy of $\mathcal{N}$, and monotonicity of $\mathcal{N}$ yields the result. The simplicity of zeroes again follows from Proposition 1.1. □

Lemma 2.3 *If $\alpha < \tilde{\alpha}$ are real parts of σ and $\tilde{\sigma}$ respectively, then one has $k \geq \tilde{k}$ for the values of $\mathcal{N}$ on the spaces P_α and $P_{\tilde{\alpha}}$.*

Proof. Let $x_0 \in P_\alpha \setminus 0, \tilde{x}_0 \in P_{\tilde{\alpha}}$ and let $x(t)$, $\tilde{x}(t)$ denote solutions through these points. Let $y(t) = x(t) + \tilde{x}(t)$. One has $x(t) = e^{\alpha t} t^a q(t) + O(t^{a-1} e^{\alpha t})$, $\tilde{x}(t) = e^{\tilde{\alpha} t} t^{\tilde{a}} \tilde{q}(t) + O(t^{\tilde{a}-1} e^{\tilde{\alpha} t})$ for $\alpha < \tilde{\alpha}$ and quasiperiodic $q(t)$ and $\tilde{q}(t)$.

Observing that $e^{-\alpha t} t^{-a} y(t) - q(t) \to 0$ as $t \to -\infty$, one has as in the lemma above that

$$\mathcal{N}(y(t)) = \mathcal{N}(e^{-\alpha t} t^{-a} y(t)) = \mathcal{N}(q(t)) = \mathcal{N}(x(t)) = k$$

for t arbitrarily close to $-\infty$. Similiary, one has $\mathcal{N}(y(t)) = \tilde{k}$ for t close to ∞. Thus $k \geq \tilde{k}$ follows from the monotonicity of $\mathcal{N}$. □

Lemma 2.4 *Let $S = \{\alpha \,|\, \alpha = Re\ \sigma$ where σ is an eigenvalue of (12) $\}$. If the value $N = k$ on P_α is independent of $\alpha \in S$, then $\mathcal{N}(x(t)) \equiv k$ for any nontrivial solution with initial condition $x_0 \in span_{\alpha \in S} P_\alpha$.*

Proof. Let us write $x(t) = \sum_{\alpha \in S} x_\alpha(t)$ as a finite sum of solutions with initial conditions in each P_α. Then we can argue as in the proof of the previous Lemma to show that

$$\lim_{t \to \pm\infty} \mathcal{N}(x(t)) = k.$$

Thus $\mathcal{N}(x(t)) = k$ for all t by the monotonicity of $\mathcal{N}$. □

Lemma 2.5 *If S is as in 2.4, then*

$$dim\ span_{\alpha \in S} P_\alpha \leq 2.$$

In particular, for any α one has

$$dim\ P_\alpha \leq 2$$

Proof. Suppose the space $span_{\alpha \in S} P_\alpha$ in question has dimension three or more. Then there exists a nontrivial solution $x(t)$ with initial condition x_0 in that space with

$$x_i(0) = \dot{x}_{i-1}(0) = 0.$$

Such a solution is easily found by taking a nontrivial linear combination of three linearly independent solutions. Now $\mathcal{N}(x(t)) \equiv k$ is constant in t by Lemma 2.4; but this contradicts Proposition 1.1. □

3 The Morse Decomposition

As was mentioned in the introduction the framework of a Morse decomposition will be used to state and prove our results. The main tool in the definition of a Morse decomposition of the global attractor of $\mathcal{CFS}$ is the Lyapunov function $\mathcal{N}$ defined in the introduction.

To simplify the notation, we will let

$$\dot{x} = C(x) \tag{13}$$

denote the $\mathcal{CFS}$ satisfying (1) and refer to this system as "system C".

Definition 3.1 Throughout this section J will represent the number of the eigenvalues with positive real part for the linearization of system C at $x = 0$.

It will be assumed that $J > 0$, otherwise the Morse decomposition will be trivial and Theorem 1.4 and 1.5 are trivially satisfied.

Definition 3.2 Assume $0 \leq i < n$.

$$\begin{aligned}
\text{If } \Delta = -1 \text{ and } n \text{ is odd, then} \quad P &= \begin{cases} \frac{n+1}{2} & \text{if } J = n \\ i-1 & \text{if J=2i,2i+1.} \end{cases} \\
\text{If } \Delta = 1 \text{ and } n \text{ is odd, then} \quad P &= \begin{cases} \frac{n+1}{2} & \text{if } J = n \\ i-1 & \text{if J=2i-1,2i.} \end{cases} \\
\text{If } \Delta = -1 \text{ and } n \text{ is even, then} \quad P &= \begin{cases} \frac{n}{2} & \text{if } J = n \\ i-1 & \text{if J=2i,2i+1.} \end{cases} \\
\text{If } \Delta = 1 \text{ and } n \text{ is even, then} \quad P &= \begin{cases} \frac{n+2}{2} & \text{if } J = n \\ i-1 & \text{if J=2i-1,2i.} \end{cases}
\end{aligned}$$

As will be seen $P + 1$ is the number of Morse sets in the Morse decomposition alluded to in the Introduction.

Definition 3.3

$$\begin{aligned}
\text{If } \Delta = -1 \text{ then } \tilde{M}(p) &= \{x(t) : \mathcal{N}(x(t)) = 2p + 1 \text{ for all } t\}. \\
\text{If } \Delta = 1 \text{ then } \tilde{M}(p) &= \{x(t) : \mathcal{N}(x(t)) = 2p \text{ for all } t\}.
\end{aligned}$$

Set

$$M(p) = \tilde{M}(p) \qquad \text{for } p = 0, \ldots, P-1$$

and

$$M(P) = \{0\} \cup \bigcup_{i>P} \tilde{M}_i.$$

If $J = n$ then the last equation is to be understood as $M(P) = \{\mathbf{0}\}$. *Observe that $\tilde{M}_i$ is empty for sufficiently large i .*

Proposition 3.4 *The collection*

$$\mathcal{M}(\mathcal{A}) = \{M(p) \mid p = 0, \ldots, P\}$$

is a Morse decomposition of the global attractor $\mathcal{A}$ *with an admissible ordering* $p > p-1$

Proof. It needs to be shown that if $x \in \mathcal{A}$ then there are $L \geq D$ such that the alpha and omega limit sets of x satisfy $\alpha(x) \subset M(L)$ and $\omega(x) \subset M(D)$. We begin by proving that there is a D such that $\omega(x) \subset M(D)$.

Since $\mathcal{A}$ is an isolated invariant set $\omega(x) \subset \mathcal{A}$. Now assume that there exist $z, y \in \omega(x)$ such that

$$\mathcal{N}(z) > \mathcal{N}(y).$$

Then, there are sequences $\{t_n\} \to \infty$ and $\{s_n\} \to \infty$ such that

$$x(t_n) \to z \ , \ x(s_n) \to y.$$

Since the number $\mathcal{N}$ is defined on an open set and is locally constant for t_n and s_n sufficiently large, one can find subsequences (denoted again by $\{t_n\}$, $\{s_n\}$) such that

$$t_n < s_n < t_{n+1} < s_{n+1} < \ldots$$

and $\mathcal{N}(x(t_n)) = \mathcal{N}(z)$ for all n and $\mathcal{N}(x(s_n)) = \mathcal{N}(y)$ for all n. This contradics the monotonicity of $\mathcal{N}$ i.e. Proposition 1.1c. Thus $D = \mathcal{N}(z) = \mathcal{N}(y)$.

The same argument applies to $\alpha(x)$ for $t \to -\infty$, giving rise to L such that $\alpha(x) \subset M_L$. By Proposition 1.1 $L \geq D$.

The second property of the Morse decomposition follows directly from Definition 3.3. □

Definition 3.5 An isolating neighborhood W is an *isolating block* if the function $\omega : W \to [0, \infty]$ defined by

$$\omega(x) = \sup\{t > 0 \mid x \cdot [0, t] \subset W \text{ for } x \in W\}$$

is continuous.

We shall now build specific isolating blocks for the Morse sets. Since we assume that there exists a global attractor $\mathcal{A}$ for the system C, there exists a bounded set K such that $\mathcal{A} \subset K$.

Definition 3.6 For $p = 0, \ldots, P-1$ and $\Delta = -1$ let

$$\tilde{W}(p) = \{\bigcup Q(\sigma_1, \ldots, \sigma_n) \cap K \mid \text{ if } x \in Q(\sigma_1, \ldots, \sigma_n) \text{ then } \mathcal{N}(x) = 2p+1\}.$$

and for $\Delta = 1$ let

$$\tilde{W}(p) = \{\bigcup Q(\sigma_1, \ldots, \sigma_n) \cap K \mid \text{ if } x \in Q(\sigma_1, \ldots, \sigma_n) \text{ then } \mathcal{N}(x) = 2p\}.$$

Observe that $\tilde{W}(p)$ is the union of all "orthants" on which the value of $\mathcal{N}$ is prescribed constant, restricted to K. $\tilde{W}(p)$ is not an isolating neighborhood for $M(p)$ since $0 \in \tilde{W}(p)$. Let

$$W(P) := B_\epsilon \cup (K \setminus \bigcup_{i=1}^{P-1} \tilde{W}(i))$$

where B_ϵ is the closed ϵ-ball centered at the origin and ϵ is sufficiently small.

Finally, define

$$W(p) = cl(\tilde{W}(p) \setminus W(P)) \ , \ p = 0, \ldots, P-1$$

where $cl\ A$ denotes the closure of A.

Before proving that $W(p)$ is an isolating neighbourhood for $M(p)$ we need the following lemma.

Lemma 3.7 *If the origin in the system C possess a nontrivial center manifold, then there exists an integer k_0 and a quantity $\epsilon > 0$ such that for any sufficiently small neigborhood U of the origin one has*

$$x \in X \text{ and } \mathcal{N}(x) = k_0$$

whenever $x \in U$ satisfies

$$x \neq 0, \text{ and } dist(x, W^c) \leq \epsilon |x|.$$

Proof. Suppose the lemma is false. Then there exists a sequence $x_m \to 0$ such that $x_m \neq 0$ and $dist(x_m, W^c) \leq \epsilon_m |x|$ for some $\epsilon_m \to 0$ satisfing for each m either $x_m \notin X$ or $x_m \in X$ but $\mathcal{N}(x_m) \neq k_0$. Let $z_m = \frac{x_m}{|x_m|}$. Then $z_m \to z_0 \in T_0 W^c$ where $T_0 W^c$ is the tangent space to the center manifold at the origin. Observe that $z_0 \neq 0$. Therefore, $z_0 \in P_0 - \{0\}$, where P_0 is the subspace coresponding to the zero eigenvalue of the linearization.

By Lemma 2.2, $z_0 \in X$ and $\mathcal{N}(z_0) = k_0$. Thus for the large m, $z_m \in X$ and hence $x_m \in X$. This implies that

$$\mathcal{N}(x_m) = \mathcal{N}(z_m) = \mathcal{N}(z_0) = k_0$$

a contradiction. □

Now we can prove the following theorem.

Theorem 3.8 *There is an ϵ such that for all p, $W(p)$ is an isolating block for $M(p)$.*

Proof. It needs to be shown that $M(p) \cap \partial W(p) = \emptyset$ where $\partial W(p)$ is the boundary of $W(p)$.

Since $\mathcal{A}$ is a global attractor we can assume the set K was chosen such that if $x_0 \in \partial K$ then $x_0 \cdot [0, \infty) \subset K$.

If the origin $\mathbf{0}$ is hyperbolic in system C, then there is an ϵ sufficiently small such that the ϵ-ball is an isolating neighborhood for the origin.

If the origin is not hyperbolic, then by the Lemma 3.7 there is an B_ϵ about the origin, such that the recurrent set, which contains the origin in its closure, lies in $B_\epsilon \cap (K \setminus \bigcup_{i=1}^{P-1} W(k))$, thus in $W(P)$.

By the Proposition 1.1 *d.* every orbit upon coming out of some $W(k)$ immediately enters $W(j)$, $k > j$ and by the monotonicity of $\mathcal{N}$ cannot come back. Internal tangencies to $\partial W(k)$ are, therefore, ruled out. □

4 The Conley Indices

Our computation of the Conley indices of the Morse sets takes advantage of the continuation properties of the index and the fact that, under the assumption of the existence of a global attractor, the only bifurcations which effect the indices are bifurcations from the origin. To understand these bifurcations we start with a fixed $\mathcal{CFS}$ which possesses an extremely simple linear form and study the affect of homotopies from this system. In particular, by carefully controlling the non-linearity we construct a homotopy along which the bifurcations at the origin are super-critical Hopf or pitchfork bifurcations. The Conley index of each Morse set $M(p)$ is then computed near the bifurcation point. Finally, a homotopy to the system C which preserves the Morse decompostion is constructed. Thus, throughout this section we think of system C as being a fixed $\mathcal{CFS}$, the global dynamics of which we wish to understand.

Let us begin by considering the $n \times n$ matrix

$$L(s) = \begin{pmatrix} -s & 0 & 0 & \cdots & \pm 1 \\ 1 & -s & 0 & \cdots & 0 \\ 0 & 1 & -s & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -s \end{pmatrix}$$

Observe that

$$\dot{x} = L(s)x \tag{14}$$

is an n-dimensional $\mathcal{CFS}$ with $\Delta = \pm$ determined by the choice of the upper right hand entry of L. Given system C we choose L such that n and Δ agree for (13) and (14).

The eigenvalues for $L(s)$ will be denoted by

$$\sigma_i(s) = \alpha_i(s) + \imath\beta_i(s) \qquad i = 1, \ldots, n.$$

As the following lemma indicates these eigenvalues can be computed explicitly.

Lemma 4.1 *The real parts of the eigenvalues of $L(s)$ are ordered as follows:*

$$\begin{array}{lllll} if & \Delta = -1 & n = odd & then & \alpha_1(s) = \alpha_2(s) > \alpha_3(s) = \cdots > \alpha_{n-2}(s) = \alpha_{n-1}(s) > \alpha_n(s) \\ if & \Delta = -1 & n = even & then & \alpha_1(s) = \alpha_2(s) > \alpha_3(s) = \cdots > \alpha_{n-1}(s) = \alpha_n(s) \\ if & \Delta = 1 & n = odd & then & \alpha_1(s) > \alpha_2(s) = \alpha_3(s) > \cdots > \alpha_{n-1}(s) = \alpha_n(s) \\ if & \Delta = 1 & n = even & then & \alpha_1(s) > \alpha_2(s) = \alpha_3(s) > \cdots > \alpha_{n-2}(s) = \alpha_{n-1}(s) > \alpha_n(s) \end{array}$$

Proof. Given $\Delta = \pm 1$, the characteristic polynomial for $L(s)$ is

$$(x+s)^n = \pm 1.$$

□.

Now, let

$$\dot{x} = G(\lambda, x) \qquad \lambda \in [0,1]$$

denote an arbitrary 1-prameter family of equations in the class of $\mathcal{CFS}$ such that $dG(0,0) = L(2)$. Let $L' = dG(1,0)$.

Lemma 4.2 *The real parts of the eigenvalues of L' are ordered as follows:*

$$\begin{array}{lllll} if & \Delta = -1 & n = odd & then & \alpha_1(s) \geq \alpha_2(s) > \alpha_3(s) \geq \cdots > \alpha_{n-2}(s) \geq \alpha_{n-1}(s) > \alpha_n(s) \\ if & \Delta = -1 & n = even & then & \alpha_1(s) \geq \alpha_2(s) > \alpha_3(s) \geq \cdots > \alpha_{n-1}(s) \geq \alpha_n(s) \\ if & \Delta = 1 & n = odd & then & \alpha_1(s) > \alpha_2(s) \geq \alpha_3(s) > \cdots > \alpha_{n-1}(s) \geq \alpha_n(s) \\ if & \Delta = 1 & n = even & then & \alpha_1(s) > \alpha_2(s) \geq \alpha_3(s) > \cdots > \alpha_{n-2}(s) \geq \alpha_{n-1}(s) > \alpha_n(s) \end{array}$$

Proof. By Lemma 4.1, the ordering of the real parts of the eigenvalues for $L(2)$ is known. Assume that $\alpha_i(2) = \alpha_{i+1}(2)$. By Lemma 2.3, $\mathcal{N}$ assumes a unique value on the two dimensional plane (take away the origin) spanned by the eigenvectors corresponding to eigenvalues $\sigma_i(2) = \sigma_{i+1}(2)$. $\sigma_k(s)$, and hence, $\alpha_k(s)$ depends continuously on s, but $\mathcal{N}$ is locally constant, thus $\mathcal{N}$ remains constant on the planes spanned by $\sigma_i(s)$ and $\sigma_{i+1}(s)$. Finally, since Lemma 4.1 states that

$$\alpha_{i-1}(2) < \alpha_i(2) = \alpha_{i+1}(2) < \alpha_{i+2}(2),$$

the proof follows from Lemma 2.5. □

Remark 4.3 We need inequalities in the statement of the Lemma 4.2 as opposed to the equalities in the Lemma 4.1, because during a general homotopy in the class of $\mathcal{CFS}$ pairs of complex eigenvalues may become pairs of distinct real eigenvalues.

In order to obtain a global attractor we consider the nonlinear $\mathcal{MCFS}$

$$\dot{x} = L(2) - h(x) =: T(x) \tag{15}$$

where $h(x) = (x_1^3, \ldots, x_n^3)$.

Lemma 4.4 *The origin* $\mathbf{0}$ *is the global attractor for (15).*

Proof. Let

$$E = \sum_i^n x_i^2.$$

Then E is a Lyapunov function for (15) which is stricly decreasing off the origin. □

Given system C, recall the meaning of J (Definition 3.1) and P (Definition 3.2). Choose $u \in \mathbf{R}$ such that $L(u)$ has:

1. $2P$ $(\Delta = -1)$ or $2P + 1$ $(\Delta = 1)$ eigenvalues with positive real parts if $J < n$ or
2. n eigenvalues with positive real parts if $J = n$.

Lemma 4.1 shows that such a number u exists.

Definition 4.5 Let $F(s, x)$ be a homotopy

$$\dot{x}(s,t) = F(s,x) = L(s(u-2)+2)x - h_s(x)$$

where $h_0(x) = h(x)$ and $h_s(x) \sim O(x^2)$ at $x = 0$.

Observe that $F(0, x) = T(x)$ and the linear part of $F(s, x)$ at $x = 0$ is $L(s(u-2)+2)$. Thus, by Lemma 4.1 we can conclude that the set of parameter values at which the origin undergoes a bifurcation is of the form $\{s_p \mid p = 0, \ldots, P-1\}$ where

$$0 < s_0 < s_1 < \ldots < s_{P-1} < 1.$$

Again, by Lemma 4.1, if $J < n$ for $1 \le p \le P-1$, Hopf bifurcations occur, while at s_0 either a Hopf bifurcation $(\Delta = -1)$ or a pitchfork bifurcation $(\Delta = 1)$ occurs.

If $J = n$ a similar argument shows that Hopf bifurcations occur at s_p for $1 \le p \le P-2$. while at $p = 0$ or $p = P-1$ either Hopf or pitchfork bifurcations occur, depending on Δ and the parity of n.

Theorem 4.6 *There exists a one-dimensional family of nonlinearities $h_s(x)$ such that all the aforementioned Hopf bifurcations are nondegenarate and supercritical.*

The proof of this Theorem is technical in nature and has no bearing on the rest of the presentation, hence we relegate it to Section 8. However, it is the key to the proof of the following essential lemma.

Lemma 4.7 *The cohomological Conley indices of the Morse sets for the system $\dot{x} = F(1, x)$ are as follows.*

$$CH^k(M(P)) = \begin{cases} \mathbf{Z} & k = 2P \\ 0 & otherwise \end{cases}$$

If $J < n$ and $\Delta = 1$, then

$$\begin{aligned} CH^k(M(0)) &= \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & k = 0 \\ 0 & otherwise \end{cases} \\ CH^k(M(p)) &= \begin{cases} \mathbf{Z} & k = 2p-1, 2p \\ 0 & otherwise \end{cases} \qquad p = 1, \ldots, P-1. \end{aligned}$$

If $J < n$ and $\Delta = -1$, then

$$CH^k(M(p)) = \begin{cases} \mathbf{Z} & k = 2p, 2p+1 \\ 0 & otherwise \end{cases} \qquad p = 0, \ldots, P-1.$$

If $J = n$, then the indices of the Morse sets $M(p)$, $p \neq P-1$ are as above. The remaining index is as follows.
If $\Delta = 1$ and n is even or if $\Delta = -1$ and n is odd, then

$$CH^k(M(P-1)) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & k = n-1 \\ 0 & otherwise. \end{cases}$$

If $\Delta = 1$ and n is odd or if $\Delta = -1$ and n is even, then

$$CH^k(M(P-1)) = \begin{cases} \mathbf{Z} & k = n-2, n-1 \\ 0 & otherwise. \end{cases}$$

Proof. $CH^*(M(P))$ is computed directly from the linearization about $\mathbf{0}$.

For $p \neq P$, let $\tilde{M}^s(p)$ denote the Morse set of Definition 3.3 for the system $\dot{x} = F(s, x)$ and similarly for $M^s(p)$. Observe that $\tilde{M}^0(p) = \emptyset$ for $p > 0$ and hence $CH^*(\tilde{M}^0(p)) = 0$. For $0 \leq s < s_p$, the unstable set of the origin $\mathbf{0}$ does not intersect $\tilde{W}(p) \setminus \{0\}$, and hence, $(0, \tilde{M}^s(p))$ is an attractor repeller pair for $\mathrm{Inv}\tilde{W}(p)$. By Lemma 3.7 $(0, \tilde{M}^s(p))$ remains an attractor repeller pair when $s = s_p$. This implies that $\tilde{M}^s(i)$ is bounded away from $\mathbf{0}$ by some $\nu > 0$, i.e. $x \in \tilde{M}^s(p)$ implies that $||x|| > \nu$. Since attractor repeller pair decompostions are stable under perturbation, for $\mu > 0$ but sufficiently small, if $s_p < s < s_p + \mu$, then there exists an attractor repeller pair decomposition (A^s, R^s) of $\mathrm{Inv}\tilde{W}(p)$ such that $x \in A^s$ implies that $||x|| < \nu/2$ and $x \in R^s$ implies that $||x|| > \nu/2$. Observe that R^s continues to $\tilde{M}^{s_p}(p)$ as $s \to s_p$ and hence

$$CH^*(R^s) \approx CH^*(\tilde{M}^{s_p}(p)) \approx 0.$$

Choosing ν sufficiently small, the dynamics in $\{x \in \tilde{W}(p) \mid ||x|| \leq \nu/2\}$ is completely determined by the nondegenerate Hopf or pitchfork bifurcation. Since, by Theorem 4.6, it is a supercritical bifurcation we can form an attractor repeller decomposition of A^s given by $(\Gamma_p^s, 0)$ where Γ_p^s is either a hyperbolic periodic orbit or a pair of hyperbolic fixed points of the same index.

At this point we have shown that

$$\{R^s, 0, \Gamma_p^s\}$$

forms a Morse decomposition of $\text{Inv}(\tilde{M}^s(p))$ for $s_p < s < s_p + \mu$. Furthermore, after the bifurcation, i.e. for $s > s_p$ on the stable manifold of the origin $\mathcal{N}$ takes values greater than on $W(p) \setminus \{0\}$. Thus there exist no connecting orbits from R^s to 0. Therefore,

$$0 > R^s > \Gamma_p^s$$

is an admissible ordering for the Morse decomposition. This implies that for $\mu > 0$ sufficiently small (Γ_p^s, R^s) is an attractor repeller pair for $\text{Inv}W(p) \approx M^s(p)$. Since $CH^*(R^s) \approx 0$, $CH^*(M^s(p)) \approx CH^*(\Gamma_p^s)$, which give the desired result for $M(p)$, $0 \leq p < P$. □

Corollary 4.8 *For each $p < P$, there exists $\mu_p > 0$ such that for $s = s_p + \mu_p$, $M^s(p)$ admits an attractor repeller decomposition (Γ_p^s, R_p^s) where $CH^*(R_p^s) \approx 0$ and Γ_p^s is a hyperbolic periodic orbit or two hyperbolic fixed points.*

Having computed the Conley indices for the Morse decomposition of $\dot{x} = F(1, x)$, we now want to continue to the system C. In order to know that the Morse decomposition continues we need to find a homotopy in the class of $\mathcal{CFS}$ for which the isolating neighborhoods $W(p)$, $p = 0, \ldots, P$ are preserved.

Theorem 4.9 *There exists a homotopy $E(s, x)$ such that:*

1. *$E(0, x) = F(1, x)$ and $E(1, x) = C(x)$;*
2. *$E(s, x)$ is the cyclic feedback system for all s;*
3. *there is no bifurcation from the origin into the spaces $W(0), \ldots, W(P-1)$ during the homotopy.*

Corollary 4.10 *The Conley indices of the Morse sets in the Morse decomposition of system C are the same as those of Lemma 4.7.*

Proof of Corollary 4.10. It need to be shown that the neighborhoods $W(p)$ $p = 0, \ldots, P$ isolate throughout the homotopy $E(s, x)$. Let

$$M^s(p) := \text{Inv}W(p)$$

in the system $\dot{x} = E(s, x)$. Theorem 4.9.2 implies that it suffices to show that $M^s(p) \cap B_\epsilon = \emptyset$ for $p = 0, \ldots, P-1$ and $s \in [0, 1]$.

Since s is running through a compact set $\bigcup_{s\in[0,1]} M^s(p)$ is a compact set for all p. Furthermore

$$\bigcup_{s\in[0,1]} M^s(p) \subset \tilde{W}(p)$$

for all p. Thus by Theorem 4.9.3 there exists an ϵ_0 such that for a ball B_{ϵ_0} around the origin

$$B_{\epsilon_0} \cap \bigcup_p \bigcup_{s\in[0,1]} M(p)^s = \emptyset.$$

We may assume without loss of generality that $\epsilon < \epsilon_0$ which finishes the proof. □

The proof of Theorem 4.9 occupies the rest of this section.

We can rewrite system C as

$$\dot{x} = C(x) = Dx + g(x)$$

where $D = df(0)$.

Let $E(s,x) = E_2(s,x) \circ E_1(s,x)$ be a composition of two homotopies,

$$\begin{aligned} E_1(s,x) &:= U(s) + h_1(x) \qquad & s \in [0,1] \\ E_2(s,x) &:= Dx + T(s,x) \qquad & s \in [0,1] \end{aligned}$$

where $U(0) = L(u)$, $U(1) = D$, $T(0,x) = g(x)$ and $T(1,x) = h_1(x)$. Note that in the first homotopy we change the linear part and in the second the nonlinear part . While we have to construct the first in such a way that no new bifurcations from the origin occur, the second must be constructed in a way that there are no bifurcations from infinity.

We will first show that $E_1(s,x)$ exists. We need to find a continuous family of matrices $U(s)$ within the class of linear $\mathcal{CFS}$ such that a sufficient portion of the spectrum lies in the right half plane. We will construct the family $U(s)$ in several steps.

Let

$$D = \begin{pmatrix} c_1 & 0 & 0 & \cdots & \Delta a_1 \\ a_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & c_3 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & a_n & c_n \end{pmatrix}$$

where $\Delta = \pm 1$. Let $c = \min\{c_i\}$. Define

$$G_1(s) = \begin{pmatrix} c_1 + s(\gamma - c_1) & 0 & 0 & \cdots & \Delta a_1 \\ a_2 & c_2 + s(\gamma - c_2) & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & a_n & & c_n + s(\gamma - c_n) \end{pmatrix}$$

where γ satisfies

$$c + \gamma > \Pi_{i=1}^n a_i. \tag{16}$$

Let $\bar{D} = G_1(1)$ and let $d_i = c_i + \gamma$.

Lemma 4.11 *Let $\lambda_j = \mu_j + i\nu_j$ $j = 1, \ldots, n$ be the eigenvalues of D and let $\bar{\lambda}_j = \bar{\mu}_j + i\bar{\nu}_j$ $j = 1, \ldots, n$ be the eigenvalues of $\bar{D}$. Then $\bar{\mu}_j = \mu_j + \gamma$ and $\bar{\nu}_j = \nu_j$ for all j.*

Proof. The eigenvalues λ_i are the roots of the characteristic polynomial

$$\prod_{k=1}^{n}(\lambda - c_k) = \Delta \prod_{k=1}^{n} a_k. \tag{17}$$

Let us fix one eigenvalue $\lambda_i =: \lambda$. Set $\lambda - c_k = |m_k|e^{i\alpha_k}$. Then (17) becomes

$$(\prod_{k=1}^{n}|m_k|)e^{i\sum\alpha_k} = \Delta\prod_{k=1}^{n} a_k$$

and so we have

$$e^{i\sum_k \alpha_k} = \Delta \tag{18}$$

$$\prod_j |m_k| = |\prod_k a_k|. \tag{19}$$

Thus for

$$\Delta = 1 \text{ we have } \sum_k \alpha_k = 2\pi T, \quad T = 1, \ldots$$

and for

$$\Delta = -1 \text{ we have } \sum_k \alpha_k = 2\pi T, \quad T = 0, \ldots.$$

The geometric interpretation of α_k is as follows. The angle α_k is the angle between the real axis and the half line $\overrightarrow{c_k\lambda}$. If λ is real, then $\alpha_k = 0$ for $c_k > \lambda$ and $\alpha_k = 2\pi$ for $c_k < \lambda$.

With this description of the eigenvalues we see that by moving the diagonal entries in the matrix D and keeping the off diagonal entries intact we change neither m_k nor α_k $k = 1, \ldots, n$. In other words the effect of the homotopy $G_1(s)$ is just to slide the whole spectrum to the right by γ.

Note also that by (16) all the eigenvalues of $\bar{D}$ have positive real part. □

Define

$$G_2(s) = \begin{pmatrix} (1-s)d_1 + sQ & 0 & 0 & \cdots & \Delta a_1 \\ a_2 & (1-s)d_2 + sQ & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & a_n & & (1-s)d_n + sQ \end{pmatrix}$$

where $Q = \frac{\sum_{k=1}^{n} d_k}{n}$. Let $\tilde{D} := G_2(1)$. Observe that the characteristic polynomial of $\tilde{D}$ is

$$(\lambda - Q)^n = \Delta \prod a_k$$

and so the eigenvalues are placed on a circle of the radius $\prod a_k$. Furthermore, since $Q \geq c_l + \gamma$, by (16) no eigenvalue crosses the imaginary axis during this homotopy.

Let

$$G_3(s) = \begin{pmatrix} Q & 0 & 0 & \cdots & \Delta(1-s)a_1 + \Delta \\ (1-s)a_2 + s & Q & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & (1-s)a_n + s & & Q \end{pmatrix}$$

and let $D' := G_3(1)$. Observe that during this homotopy we are shrinking the radius of the circle on which the eigenvalues lie. Also, observe that in fact

$$D' = L(Q).$$

Let $G_4(s) = L((1-s)Q + su)$.

Let us denote the eigenvalues of $\mathcal{G}(s) := G_4 \circ G_3 \circ G_2 \circ G_1$ (we rescale the homotopy parameter s in such a way that $s \in [0,1]$) by

$$\sigma_k(s) = \alpha_k(s) + i\beta_k(s).$$

Since $\mathcal{G}(s)$ is the $\mathcal{CFS}$ for all s, by Lemma 4.2 we can order the eigenvalues with respect to the real part

$$\alpha_1(s) \geq \alpha_2(s) > \alpha_3(s) \geq \cdots > \alpha_{n-2}(s) \geq \alpha_{n-1}(s) > \alpha_n(s).$$

By construction of the homotopy $\mathcal{G}(s)$ we have that the only bifurcations from the origin takes place in $\tilde{W}(p)$, $p > P-1$ and hence in $W(P)$ so condition (3) is satisfied.

Now we turn to the homotopy $E_2(s,x)$.

Let A_1 be a global attractor of $\dot{x} = Dx + h_1(x)$ and let A_0 be a global attractor of $\dot{x} = Dx + g(x) = C(x)$.

Lemma 4.12 *There exists a homotopy $E_2(s,x) = Dx + T(s,x)$ with $T(0,x) = g(x)$, $T(1,x) = h_1(x)$ and a bounded set U such that*

$$A(s) \subset U$$

where $A(s)$ is a continuous family of invariant sets with $A(0) = A_0$ and $A(1) = A_1$.

Proof. Since the entries of D are bounded, there exists k such that the vector field $\psi(x,t)$ generated by

$$\dot{x} = Dx + h_1(x)$$

points inward on the boundary of the ball $B_l := \{x \in \mathbf{R}^n \mid ||x|| \leq l\}$ for all $l \geq k$. Since A_0 is compact there is a constant $p \geq k$ such that $A_0 \subset B_p$. Let

$$V := cl(\varphi((0,\infty), B_{p+1})$$

where φ is a flow generated by $\dot{x} = G(x) := Dx + g(x)$.

Clearly V is positively invariant and a simple argument shows that it is bounded. Hence there is $R \geq p+1$ such that $V \subset B_R$.

Let $Bf := B_{p+1} \setminus B_p$. Bf will act as a buffer zone during the homotopy i.e. it will separate $A(s)$ from (possible) invariant sets created by bifurcations at infinity.

Let $x_0 \in Bf$. Then both $\varphi(x_0,t), \psi(x_0,t) \subset B_R$ for all $t \geq 0$ and since $A_0, A_1 \subset B_p$ for each x_0 there is a time t_0 such that $\varphi(x_0,t) \subset B_p$ and $\psi(x_0,t) \subset B_p$ for all $t \geq t_0$. Since Bf is compact there is a uniform time $T > 0$ such that

$$\varphi(x_0,t) \subset B_p, \ \psi(x_0,t) \subset B_p$$

for all $x_0 \in Bf$ and all $t \geq T$.

Let $Y(s,x) := sg(x) + (1-s)h_1(x), s \in [0,1]$. Let us consider homotopy

$$T_1(\lambda, s, x) = \begin{cases} Y(s,x) & x \notin B_R \\ g(x) & x \in B_p \\ \lambda Y(s,x) + (1-\lambda)g(x) & \lambda = \frac{||x||-p}{R-p}, \quad ||x|| \in [p,R] \end{cases}.$$

This homotopy is a combination of two homotopies: $Y(s,x)$ which changes function g into function h_1 in the neighborhood of infinity and the identity acting on B_p. In the region $B_R \setminus B_p$ $T_1(\lambda, s, x)$ is a a linear combination of the two.

Our next goal is to show that the region Bf acts as a buffer zone. We will show that for any s the solution $y(t)$ of

$$\dot{y} = Dy + T_1(\lambda, s, y)$$

with initial data $y(0) \in Bf$ enters the ball B_p in a finite time T, i.e.

$$y(t) \in B_p \text{ for all } t \geq T. \tag{20}$$

Let us first consider the flow generated by

$$\begin{aligned} \dot{w} = H(w) &:= Dw + (1-s)h(w) + sg(w) = (1-s)[Dw + h(w)] + s[Dw + g(w)] \\ &= (1-s)F(w) + sG(w). \end{aligned}$$

Since both flows ψ and φ have the property (20) and B_p is convex, the flow generated by (21) also has the property (20).

Now consider $\dot{y} = Dy + T_1(\lambda, s, x)$ with $y(0) \in Bf$. Observe that

$$\begin{aligned} \dot{y} &= Dy + T_1(\lambda, s, y) \\ &= Dy + \lambda Y(s,x) + (1-\lambda)g(x) \\ &= Dy + \lambda(y(t))[(1-s)h(y(t)) + sg(y(t))] + (1-\lambda(y(t)))g(y(t)) \\ &= \lambda(t)[Dy + (1-s)h(y(t)) + sg(y(t))] + (1-\lambda(t))[Dy + g(y(t))] \\ &= \lambda(t)H(y(t)) + (1-\lambda(t))G(y(t)) \end{aligned}$$

for $t \in [0, \tau)$ for some $\tau > 0$ a time when $y(t)$ leaves $B_R \setminus B_p$.

Now

$$y(t) = y(0) + \int_0^t \lambda(u)H(y(u))du + \int_0^t (1-\lambda(u))G(y(u))du.$$

This shows that $y(t)$ is a linear combination (which does depend on t) of solutions $x(t)$ and $z(t)$ with $x(0) = y(0) = z(0)$ which solve

$$\dot{x} = H(x) \qquad \text{and} \qquad \dot{z} = G(z)$$

respectively. Since B_R is convex the trajectory $y(t)$ will never leave B_R. Since both flows generated by above equations have the property (20), the flow generated by (21) has this property too. This shows that $\mathrm{Inv}(Bf)$ is empty throughout the homotopy $T_1(\lambda, s, x), s \in [0,1]$, because every trajectory $y(t)$ with initial data $y(0) \in Bf$ satisfies (20). Thus there is a continuous function $\bar{A}(s)$ with $\bar{A}(0) = A(0)$ and $A(s) \subset B(p)$ for all $s \in [0,1]$.

Now we define the second part of the homotopy by

$$T_2(s,x) := \begin{cases} sg(x) + (1-s)T_1(\lambda, 1, x) & x \in B_R \\ h_1(x) & x \notin B_R \end{cases} .$$

Since one knows the nonlinearity $h_1(x)$ explicitly it is easy to see that during this homotopy the global attractor remains bounded in B_R. Desired homotopy is now

$$T(s,x) = T_2(s,x) \circ T_1(s,x).$$

□

We have constructed desired homotopies $E_1(s,x)$, $E_2(s,x)$ and thus concluded the proof of Theorem 4.9.

5 Semiconjugacy.

In this section we will discuss the proof of Theorem 1.5.

Idea of constructing semiconjugacies of the form of Theorem 1.5 originated in the paper of McCord and Mischaikow (1992). Though the focus was on the global attractor for scalar delay equations with negative feedback, the semiconjugacy was shown to exist under rather general assumptions. As will be shown those results are immediately applicable in the case that $\Delta = -1$ and n is even. With this in mind we begin by recalling the assumptions of Theorem 1.3 of McCord and Mischaikow (1992).

A1 $\mathcal{A}$ is the global attractor for a semi-flow Φ on a Banach space. Furthermore, if φ denotes the restriction of Φ to $\mathcal{A}$ then φ defines a flow on $\mathcal{A}$.

A2 Under the flow $\varphi : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$

$$\mathcal{M}(\mathcal{A}) = \{M(p) \mid p = 1, \dots, P\}$$

with ordering $0 < \ldots < P$ is the Morse decomposition of $\mathcal{A}$.

A3 For each $p = 0, \dots, P-1$, $M(p)$ has a Poincaré section h_p defined on the neighbohood $W(p)$.

A4 The cohomology indices of the Morse sets are

$$CH^k(M(P), \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2P \\ 0 & \text{otherwise} \end{cases}$$

and for $p = 0, \dots, P-1$

$$CH^k(M(p), \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2p, 2p+1 \\ 0 & \text{otherwise .} \end{cases}$$

A5 For each $M(p)$, $p < P$, there is a continuation of the flow in a neighborhood of $M(p)$ to an isolated invariant set which admits the attractor-repellor decomposition where attractor is a hyperbolic periodic orbit and a repeller has trivial Conley index. The continuation preserves the Poincaré section.

Obviously **A1** and **A2** are satisfied for $\mathcal{CFS}^{\pm}$. Observe that in the hypotheses of Theorem 1.5 we are assuming that if $M(p)$ has the index of a hyperbolic periodic orbit then it has a Poincaré section. In other words, we are assuming that if **A4** is satisfied then so is **A3**. Furthermore, by Corollary 4.8 if the index of $M(p)$ is that of a periodic orbit, then **A5** is satisfied. If we now set $\Delta = -1$ and assume n is even or if we assume that $\Delta = -1$ and $J < n$, then **A4** is satisfied, i.e. in these cases Theorem 1.5 is proven.

Therefore, it remains to prove Theorem 1.5 when $\Delta = 1$ or $\Delta = -1$ and $J = n$ for n odd. Observe that in all of these cases either $M(0)$, $M(P-1)$ or both fail to satisfy conditions **A3**–**A5**. In particular, these Morse sets have the index of pairs of hyperbolic fixed points, not periodic orbits. At this point the semi-conjugacy constructed for Theorem 1.2 in Mischaikow (1992) becomes relevant. The hypotheses for this theorem are **A1** as above plus the following.

A2′ Under the flow $\varphi : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$

$$\mathcal{M}(\mathcal{A}) = \{M(p^{\pm}) \mid p = 0, \ldots, P-1\} \bigcup \{M(P)\}$$

with ordering $P > P - 1^{\pm} > \ldots > 1^{\pm} > 0^{\pm}$ is a Morse decomposition of $\mathcal{A}$.

A3′ The cohomology Conley indices of the Morse sets are

$$CH^k(M(P)) \approx \begin{cases} \mathbf{Z} & \text{if } k = P \\ 0 & \text{otherwise} \end{cases}$$

and for $p = 0, \ldots, P-1$

$$CH^k(M(p^{\pm})) \approx \begin{cases} \mathbf{Z} & \text{if } k = p \\ 0 & \text{otherwise} \end{cases}$$

A4′ The connection matrix for $\mathcal{M}(\mathcal{A})$ is given by

$$\Delta = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ D_0 & 0 & 0 & & \vdots \\ & D_1 & & \ddots & 0 \\ \vdots & & & 0 & 0 \\ 0 & & \ldots & D_{P-1} & 0 \end{bmatrix} \tag{21}$$

where, up to a choice of orientation, for $p = 0, \ldots, P-2$

$$D_p : CH^p(M(p^-)) \oplus CH^p(M(p^+)) \to CH^{p+1}(M(p+1^-)) \oplus CH^{p+1}(M(p+1^+))$$

is given by

$$D_p = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

and

$$D_{P-1} : CH^{P-1}(M(P-1^-))) \oplus CH^{P-1}(M(P-1^+)) \to CH^P(M(P))$$

is given by

$$D_{P-1} = [1, -1]$$

Note that rank $D_p = 1$ for all p.

In Mischaikow (1992), Theorem 1.2 the Morse sets $M(p^\pm)$ are mapped by the semiconjugacy to fixed points. This is what is done to $M(0)$ and $M(P-1)$ in Theorem 1.5 for the cases being considered here.

To indicate that the hypotheses **A3′**–**A4′** have a bearing on our problem observe that for the $\mathcal{CFS}$ if

$$CH^k(M(0)) \approx \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

then $W(0)$ actually consists of two components $W^+(0)$ and $W^-(0)$ where $W^+(0) \subset Q(+, \ldots, +)$ and $W^-(0) \subset Q(-, \ldots, -)$. Define

$$M(0^\pm) := \text{Inv}W^\pm(0)$$

in which case **A3′** is satisfied by $M(0^\pm)$. (The case of $M(P-1)$ is analogous.)

The connection matrix for system C has the same form as (21). However, for $p = 1, \ldots, P-2$, $D^p : CH^*(M(p)) \to CH^*(M(p+1))$ is given by

$$D^p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(see Mischaikow (1987) for the computation). If $M(0^\pm)$ satisfies **A3′** then up to a choice of orientation

$$D^0 : CH^*(M(0^+)) \oplus CH^*(M(0^-)) \to CH^*(M(1))$$

takes the form

$$D^0 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

This is computed, using the fact that there exists a homotopy from system C back to a parameter value where $M(0)$ appears via a nondegenerate supercritical pitchfork bifurcation from the origin $\mathbf{0}$. A similar calculation shows that if $M(P-1)$ satisfies **A3′** then

$$D^{P-2} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

and

$$D^{P-1} = [1 \ -1].$$

With this information one can now combine the techniques of the proof in McCord and Mischaikow (1992) with those in Mischaikow (1992) to complete the proof of Theorem 1.5. Perhaps it should be mentioned at this point that many of the necessary steps to complete the proof of this theorem can be found in Section 7 when the proof of Theorem 1.10 is presented for the case in which n is odd and $\Delta = 1$.

Now we proceed to prove statements which are needed in the proof of Theorem 1.8 and of part two and three of Theorem 1.4 .

Note that **A3**, on the existence of Poincaré sections for the various $M(p)$, is quite strong and often difficult to check. For $\mathcal{MCFS}$ we can replace it by the more verifyable condition of the nonexistence of fixed points.

Lemma 5.1 *Given a $\mathcal{MCFS}$, assume that $M(p)$ contains no fixed points. Let $Q = Q(\sigma_1, \ldots, \sigma_n)$ be an orthant such that $Q \cap M(p) \neq \emptyset$. Then,*

$$\mathrm{Inv} Q = \emptyset.$$

Proof. Our proof follows from Remark 3.3 of Mallet-Paret and Smith (1990)

Let $R = \mathrm{Inv} Q$. We will show that if $R \neq \emptyset$, then there exists $z = (z_1, \ldots, z_n) \in Q$ which is a fixed point for the $\mathcal{MCFS}$. Let $R_i = \pi_i R$ be the projection of R onto the (x_i, x_{i-1}) plane. Let $i^* = max\{x_i \mid x \in R\}$ and $i_* = min\{x_i \mid x \in R\}$. Since R is compact and non-empty i_* and i^* are attained.

Monotonicity of f_i in x_{i-1} implies that if we fix $x_i = \bar{x}_i$, $\bar{x}_i \in (i_*, i^*)$ then there is an $\bar{x}_{i-1}$ such that for $x_{i-1} \in (-\infty, \bar{x}_{i-1})$, $sign(\dot{x}_i(\bar{x}_i, x_{i-1})) = -\delta_i$ and for $x_{i-1} \in (\bar{x_{i-1}}, \infty)$, $sign(\dot{x}_i(\bar{x}_i, x_{i-1})) = \delta_i$. The implicit function theorem implies the existence of a smooth map

$$g_i : [i_*, i^*] \to [i_*, i^*]$$

such that $f_i(x_i, g_i(x_i)) = 0$ on $[i_*, i^*]$.

The map $g = g_2 \circ g_3 \circ \ldots g_n \circ g_1 : [1_*, 1^*] \to [1_*, 1^*]$ has a fixed point z_1 and z defined by $z_{i-1} = g_i(z_i)$, $1 \leq i \leq n$, $i \neq 2$ defines an equlibrium, with the above mentioned properties. □

The next result is interesting on its own right.

Lemma 5.2 *Given a $\mathcal{CFS}$, if $x \in X_i \cap M(p)$ and $\varphi((0,\infty), x) \cap X_i = \emptyset$ then there is a $T > 0$ and a collection $(\sigma_1, \ldots, \sigma_n)$ such that*

$$\varphi((T,\infty), x) \subset Q(\sigma_1, \ldots, \sigma_n).$$

Proof. Assume $x \in X_i \cap M(p)$ and $\varphi((0,\infty), x) \cap X_i = \emptyset$. Then there is a $\sigma_i \in \{\pm 1\}$ such that

$$\sigma_i[\varphi((0,\infty), x)]_i > 0.$$

Now $[\varphi(T_{i+1}, x)]_{i+1} = 0$ implies that there is $\sigma_{i+1} \in \{\pm 1\}$ such that

$$\sigma_{i+1}[\frac{d}{dt}(\varphi(\cdot, x)|_{T_{i+1}}]_{i+1} > 0. \tag{22}$$

Observe, that $\sigma_{i+1} = \delta_i$. Therefore there is $\epsilon > 0$ such that

$$\sigma_{i+1}[\varphi(T_{i+1} - \epsilon, x)]_{i+1} < 0 \qquad \sigma_{i+1}[\varphi(T_{i+1} + \epsilon, x)]_{i+1} > 0.$$

Since by (22) all passages through hyperplane $H : [x]_{i+1} = 0$ are in one direction, we have

$$\sigma_{i+1}[\varphi((t,\infty),x)]_{i+1} > 0 \text{ for all } t > T_{i+1}.$$

Since n is finite , by induction there is $T > 0$ and a collection $\{\sigma_k\}_{k=1}^n$ such that

$$\sigma_k[\varphi((t,\infty),x)]_k > 0$$

for all $t > T$. □

Corollary 5.3 *For any recurrent set S of $\mathcal{CFS}$ one of the following holds:*

- *$S \subset Q(\sigma_1,\ldots,\sigma_n)$ for some colection σ_i.*
- *S is large i.e. for every i there are points $y = (y_1,\ldots,y_n)$, $z = (z_1,\ldots,z_n)$ in S such that $y_i > 0$ and $z_i < 0$.*

Proof of Theorem 1.8. Let $H_i := \{x \in \mathbf{R}^n \mid \alpha_i g_i(x_i) + \beta_i f_i(x_{i-1}) = 0\}$.

Clearly $W(p) \cap X_i$ is a closed set for $p < P$ and so in order to prove that X_i is a Poincarè section we need to show that if $x \in X_i \cap M(p), p < P$ then $\varphi((0,\infty),x) \cap X_i \neq \emptyset$.

Observe, that if $\varphi((0,\infty),x) \subset Q(\sigma_1,\ldots,\sigma_n)$ then the closure of $\varphi((0,\infty),x)$ is a reccurent set S. Note, that $S \subset \text{int}Q(\sigma_1,\ldots,\sigma_n)$ or $S = \{0\}$.

But if $x \in X_i \cap M(p), p < P$ then only the former case may occur. Now $S \subset \text{int}Q(\sigma_1,\ldots,\sigma_n)$ implies

$$Q(\sigma_1,\ldots,\sigma_n) \cap H_i \neq \emptyset \qquad \text{for all } i.$$

We will show that $\prod_{i=1}^n u_i = -1$ implies that for any collection $\{\sigma_i\}$ there is a j such that

$$Q(\sigma_1,\ldots,\sigma_n) \cap H_j = \emptyset. \tag{23}$$

This will show that there is no recurrent set $S \subset \text{int}Q(\sigma_1,\ldots,\sigma_n)$ for any collection $\{\sigma_i\}$. Then Lemma 5.2 asserts that for every $x \in X_i \cap M(p)$ we have $\varphi((0,\infty),x) \cap X_i \neq \emptyset$.

Thus it is enough to show (23). A crucial observation is that for every i

$$H_i \subset \mathcal{O}_i := \bigcup Q(\sigma_1,\ldots\sigma_{i-1},\sigma_i\ldots,\sigma_n)$$

where the union is taken over all collections with $\sigma_{i-1}\sigma_i = -\alpha_i\beta_i = u_i$.

Observe that

$$\bigcap_{i=2}^n \mathcal{O}_i = Q_1 \cup Q_2$$

where $Q_1 = Q(\sigma_1,\ldots,\sigma_n), Q_2 = Q(-\sigma_1,\ldots,-\sigma_n)$ and the collection $\{\sigma_i\}$ is determined by the collection $\{u_i\}_{i=2}^n$ by the condition $\sigma_{i-1}\sigma_i = u_i$.

Now observe that $\prod_{i=1}^n u_i = -1$ implies $u_1 \neq \sigma_1\sigma_n$ which in turn implies that $\bigcap_{i=1}^n \mathcal{O}_i = \{0\}$. Thus there is no orthant Q which intersects H_i for all i. This proves (23). □

Remark 5.4 If $\sigma_1\sigma_n = u_1$ then $\bigcap_{i=1}^n \mathcal{O}_i = Q_1 \cup Q_2$ so there may be a recurrent set in $Q_1 \cup Q_2$. The remaining open question is whether it is indeed possible. As of this time we have no example of this behaviour.

6 $*$-hyperbolicity

The proof of Theorem 1.4.1 requires a slight extension of results of A. Floer (1987). Since these results also play a crucial role in the proof of Theorem 1.10 we take this opportunity to recall his work.

Let X be a locally compact space on which a flow is defined, and let T be a topological space. Let $W \subset X$ and let $\Theta : W \to T$ be a continuous function. Let M be an isolated invariant set with an attractor–repeller decomposition (A, R) and a corresponding index triple $N_0 \subset N_1 \subset N_2$ such that $N_2 \subset W$. Since $CH^*(M) \approx H^*(N_2, N_0)$, Floer observed that an operation $*_{M,\Theta}$ of $H^*(T)$ on $CH^*(M)$ could be defined, i.e.

$$H^*(T) \times CH^*(M) \to CH^*(M)$$

given by

$$(\tau, u) \mapsto \tau *_{M,\Theta} u := (\Theta|_{N_2})^* \tau \cup u.$$

Similarly, if one uses the index pairs (N_2, N_1) and (N_1, N_0) one can define operations $*_{R,\Theta}$ and $*_{A,\Theta}$.

Let $i : (N_1, N_0) \to (N_2, N_0)$ and $j : (N_2, N_0) \to (N_2, N_1)$ be inclusion maps. For any continuous map $\Theta : W \to T$ we have the following Proposition.

Proposition 6.1 *1. If $CH^*(R) = 0$, then $i^* : CH^*(M) \to CH^*(A)$ is an isomorphism and*

$$\tau *_{A,\Theta} i^*(u) = \tau *_{M,\Theta} u. \tag{24}$$

2. If $CH^*(A) = 0$, then $j^* : CH^*(R) \to CH^*(M)$ is an isomorphism and

$$\tau *_{R,\Theta} j^{*-1}(u) = \tau *_{M,\Theta} u. \tag{25}$$

Proof. Assume $CH^*(R) = 0$. The long exact sequence for the triple (N_2, N_1, N_0) implies that $i^*(u)$ is an isomorphism. Equation (24) now follows from the commutativity of the diagram

$$\begin{array}{ccc}
H^*(T) \times H^*(N_2, N_0) & \xrightarrow{id \times i^*} & H^*(T) \times H^*(N_1, N_0) \\
\downarrow \cup & & \downarrow \cup \\
H^*(N_2, N_0) & \xrightarrow{i^*} & H^*(N_1, N_0)
\end{array}$$

The proof if $CH^*(A) = 0$ is similar. □

To effectively use this operation one needs the following definition.

Definition 6.2 (Floer) A connected compact isolated invariant set M is *$*$-hyperbolic of index $n \in \mathbf{N}$* if:

(i) M is a retract of an isolating neighborhood $N \subset W$,

(ii) there exists $c \in CH^n(M)$ such that the map $H^*(M) \to CH^*(M)$ given by

$$u \mapsto u *_{M,r} c$$

is an isomorphism for any retract $r : N \to M$.

Using the isomorphism of Proposition 6.1 and the proof of Theorem 2 in Floer (1987) one obtains the following proposition.

Proposition 6.3 *Let M and M' be isolated invariant sets related by continuation such that throughout the continuation the isolating neighborhoods are contained in W. Let N_2 be an isolating neighborhood of M. Furthermore, assume that $\Theta|_{N_2} : N_2 \to T$ is a retract.*

1. *If $CH^*(R) = 0$ and A is $*$-hyperbolic, then*

$$(\Theta|_{M'})^* : H^*(T) \to H^*(M')$$

is a monomorphism.

2. *If $CH^*(A) = 0$ and R is $*$-hyperbolic, then*

$$(\Theta|_{M'})^* : H^*(T) \to H^*(M')$$

is a monomorphism.

To see how these ideas are related to $\mathcal{CFS}$ we return to the family of equations $\dot{x} = F(s, x)$ of Section 4. Fix p and recall that at the bifurcation point s_p a non–degenerate supercritical Hopf bifurcation occurs. The pair of eigenvalues which cross the imaginary axis at s_p define a two dimensional subspace of $\mathbf{R}^n$ which will be denoted by K_p. Let $\pi_p : W_p \to K_p$ be given by the restriction of an orthogonal projection of $\mathbf{R}^n$ onto K_p. Define $\Theta_p : W_p \to S^1$, the unit circle in K_p, by

$$\Theta_p(x) = \frac{\pi_p(x)}{\|\pi_p(x)\|}.$$

Since the origin is not an element of $\pi_p(W_p)$, Θ_p is a continuous map.

Let $s = s_p + \mu$ where $\mu > 0$ but sufficiently small. The fact that Γ_p^s arises from a non–degenerate Hopf bifurcation leads to two conclusions:

(i) $h_p := \Theta_p|_{\Gamma_p^s} : \Gamma_p^s \to S^1$ is a homeomorphism, and

(ii) Γ_p^s is a hyperbolic periodic orbit.

Lemma 6.4 *Γ_p^s is $*$-hyperbolic. The index of Γ_p^s is $2p$ if $\Delta = -1$ and $2p - 1$ if $\Delta = 1$.*

Proof. Since h_p is a homeomorphism

$$\lambda_p := h_p^{-1} \circ \Theta_p : W_p \to \Gamma_p^s$$

is a retract. Proposition 1 in Floer (1987) and the fact that Γ_p^s is hyperbolic guarentees that Γ_p^s is $*$-hyperbolic. The index of Γ_p^s follows from Lemma 4.7. □

Proof of Theorem 1.4.1: In the proof of Lemma 4.7 we saw that $M^s(p)$ admits an attractor repellor decomposition $\{\Gamma_p^s, R^s\}$ with $CH^*(R^s) \approx 0$. By Proposition 6.3 and Lemma 6.4 with $M = \Gamma_p^s$, $M' = M(p)$ and $T = S^1$

$$\left(\Theta|_{M(p)}\right)^* : H^*(S^1) \to H^*(M(p))$$

is a monomorphism. This can only occur if $\theta : M(p) \to S^1$ is onto. □

The remainder of this section is devoted to a preparation for the proof of Theorem 1.8. The derived result will be used in Section 7.2.

In the rest of this section let us assume that n is odd and $\Delta = 1$. Then by Lemma 6.4 there exists $c \in CH^{2p-1}(\Gamma_p^s)$ such that the map $H^*(\Gamma_p^s) \to CH^*(\Gamma_p^s)$ given by

$$u \mapsto u *_{\Gamma_p^s, \lambda_s} c$$

is an isomorphism. This allows us to define an isomorphism

$$\begin{array}{rcl} H^*(S^1) & \to & CH^*(\Gamma_p^s) \\ \tau & \mapsto & h_p^*(\tau) *_{\Gamma_p^s, \lambda_p} c = \tau *_{\Gamma_p^s, \Theta_p} c \end{array}$$

because h_p is a homomorphism. By Proposition 6.1.1

$$\tau *_{\Gamma_p^s, \Theta_p} c = \tau *_{M^s(p), \Theta_p} i^{*^{-1}} c$$

i.e. there is an isomorphism $T_0 : H^*(S^1) \to CH^*(M^s(p))$.

Observe that $W(p)$ is an isolating neighborhood for $M(p)$ throughout the homotopy E, and hence, $M(p)$ is related to $M^s(p)$ by continuation. This implies that there exists an isomorphism $\gamma : CH^*(M(p)) \to CH^*(M^s(p))$. By Theorem 1 of Floer (1987)

$$\gamma\left(\tau *_{M^s(p), \Theta_p} i^{*^{-1}} c\right) = \tau *_{M^s(p), \Theta_p} \gamma \circ i^{*^{-1}} c.$$

To simplify the notation let $d = \gamma \circ i^{*^{-1}}(c)$. Then

$$\tau \mapsto \tau *_{M(p), \Theta_p} d$$

defines an isomorphism

$$T_1 : H^*(S^1) \to CH^*(M(p)).$$

Now observe that T_1 induces an isomorphism $D_p : CH^{2p-1}(M(p)) \to CH^{2p}(M(p))$ via the composition

$$CH^{2p-1}(M(p)) \stackrel{T_1^{-1}}{\to} H^0(S^1) \stackrel{pd}{\to} H^1(S^1) \stackrel{T_1}{\to} CH^{2p}(M(p)) \tag{26}$$

where pd stands for Poincaré duality.

7 Proof of Theorem 1.8

Observe that there are four cases to Theorem 1.10 determined by the sign of Δ and the parity of n. Because the proofs are similar in each circumstance, throughout this section it will be assumed that

n *is odd* and $\Delta = +1$.

For this choice of n and Δ one can refine the Morse decomposition of $\mathcal{A}$ to obtain

$$\mathcal{M}(\mathcal{A}) = \{M(p) \mid p = 0^\pm, 1, \ldots, P\}$$

with an admissible ordering $p > p-1$ and $1 > 0^\pm$, where

$$\begin{aligned} M(0^+) &:= \operatorname{Inv} Q(+,+,\ldots,+) \setminus W(P) \\ M(0^-) &:= \operatorname{Inv} Q(-,-,\ldots,-) \setminus W(P). \end{aligned}$$

The goal is to map $\mathcal{M}(\mathcal{A})$ onto the Morse decomposition

$$\mathcal{M}(D^{2P-1}, \Psi) = \{\Pi(p) \mid p = 0^\pm, 1, 2, \ldots, P\}$$

defined in the introduction. Observe that if we let $z = (z_0, \ldots, z_{2P-2}) \in D^{2P-1}$ then

$$\begin{aligned} \Pi(0^\pm) &= (\pm 1, 0, \ldots, 0) \\ \Pi(p) &= \{z \mid z_{2p-1}^2 + z_{2p}^2 = 1\} \qquad p = 1, \ldots, P-1 \\ \Pi(P) &= \mathbf{0}. \end{aligned}$$

The intervals for the ordering of this Morse decomposition are of the form $(q, q+1, \ldots, r)$ where $0^\pm \le q \le r \le P$ or $(0^+, 0^-, 1, \ldots, r)$. For each interval the corresponding Morse set is as follows:

$$\begin{aligned} \Pi(q,\ldots,r) &= \{z \mid \sum_{p=2q-1}^{2r} z_p^2 = 1\} \quad 1 \le q \le r \le P-1 \\ \Pi(q,\ldots,P) &= \{z = (0,\ldots,0,z_{2q-1},\ldots,z_{2P-2} \mid \sum_{p=2q-1}^{2P-2} z_p^2 \le 1\} \quad 1 \le q \\ \Pi(0^\pm,\ldots,r) &= \{z \mid \sum_{p=0}^{2r} z_p^2 = 1,\ \pm z_0 \ge 0\} \quad r \le P-1 \\ \Pi(0^+,0^-,\ldots,r) &= \{z \mid \sum_{p=0}^{2r} z_p^2 = 1\} \\ \Pi(0^\pm,\ldots,P) &= \{z \mid \pm z_0 \ge 0\}. \end{aligned}$$

Observe that if $P = 0$, then the Morse decomposition $\mathcal{M}(\mathcal{A})$ is trivial. If $P = 1$, then Theorem 1.10 follows directly from Mischaikow (1992), Theorem 1.2 . Thus, from now on it will be assumed that $P \ge 2$.

The order preserving bijection between the indexing sets of the Morse decomposition $\mathcal{M}(\mathcal{A})$ and $\mathcal{M}(D^{2P-1})$ is obvious. The content of the theorem is the constuction of a continuous map

$$\rho : \mathcal{A} \to D^{2P-1}$$

such that

$$M(I) = \rho^{-1}(\Pi(I))$$

for every interval I and then demonstrating that ρ is surjective, i.e. that

$$\rho(M(I)) = \Pi(I)$$

for all I.

7.1 Constructing ρ

The construction of ρ follows the ideas presented in McCord and Mischaikow (1992) and Mischaikow (1992). One begins by defining a discontinuous map

$$\tilde{\rho} : \mathcal{A} \to \tilde{X} := S^0 \times \left(\prod_{p=1}^{P-1} S^1_p \right) \times \{P\} \times [0,1]^{P-1} \times [0,P]$$

given by

$$\tilde{\rho}(x) = (\nu(x), \phi(x), P, \tau(x), V(x))$$

where $S^0 = \{\pm 1\}$ and S^1 is the unit circle. The function ν, ϕ, τ and V will be defined later, however, ν and ϕ can be thought of as representing "angle" coordinates for the Morse sets $M(0), \ldots M(P-1)$; τ measures the "distance" from the Morse sets; V measures the "height" of the point; and P is a dummy variable which only serves to simplify the bookkeeping.

Observe that given any discontinuous function $h : Y \to \tilde{Z}$, there exists a quotient map $\eta : \tilde{Z} \to Z$ such that $\eta \circ h : Y \to Z$ is continuous. What will be shown is that there exists a quotient map

$$Q : \tilde{X} \to D^{2P-1}$$

such that $\rho := Q \circ \tilde{\rho}$ is the desired continuous function. This will be done in several steps. First, define an equivalence relation on $\mathcal{I}^{P-1} \times [0,P]$, where $\mathcal{I} = [0,1]$, as follows. Let

$$(\tau, V) = (\tau_1, \ldots, \tau_{P-1}, V) \in \mathcal{I}^{P-1} \times [0,P].$$

This notation confuses the functions τ and V with the values of their images, but does not, one hopes, confuse the reader. Define $l, r : \mathcal{I}^{P-1} \times [0,P] \to \{0, \ldots, P\}$ by

$$\begin{aligned} l(\tau_1, \ldots, \tau_{P-1}, V) &:= \begin{cases} k & \text{if } k \le V, \tau_k = 1 \text{ and for all } k < p < V,\ \tau_p \neq 1 \\ 0 & \text{if no such } k \text{ exists.} \end{cases} \\ r(\tau_1, \ldots, \tau_{P-1}, V) &:= \begin{cases} k & \text{if } V \le k, \tau_k = 1 \text{ and for all } V < p < k,\ \tau_p \neq 1 \\ 0 & \text{if no such } k \text{ exists.} \end{cases} \end{aligned}$$

Set

$$\begin{aligned} (\tau, V) \sim (\tau', V') \quad \text{if} \quad l(\tau, V) &= l(\tau', V') \\ r(\tau, V) &= r(\tau', V') \\ V &= V' \\ \tau_p &= \tau'_p \text{ for all } p \text{ such that } l(\tau, V) < p < r(\tau, V). \end{aligned}$$

Let

$$\eta_1 : [0,1]^{P-1} \times [0,P] \to Y := \frac{[0,1]^{P-1} \times [0,P]}{\sim}$$

be the induced quotient map. It is left to the reader to check that Y is homeomorphic to $\Delta^P = \langle v_0, \ldots, v_p \rangle$ the standard P simplex. Let $\lambda : Y \to \Delta^P$ be a homeomorphism such that

$$\lambda \circ \eta_1(0, \ldots, 0, \tau_k = 1, 0 \ldots, 0, k) = v_k$$

and

$$\lambda \circ \eta_1 \left(\{ (\tau, V) \mid \tau_{i_1} = \ldots = \tau_{i_n} = 0 \} \right) = \text{ face opposite } \langle v_{i_1}, \ldots, v_{i_n} \rangle.$$

Let

$$\tilde{X} := S^0 \times \left(\prod_{p=1}^{P-1} S^1_p \right) \times \{P\} \times \Delta^P$$

and let $\eta : \tilde{X} \to \hat{X}$ be given by $\eta = \text{id} \times (\lambda \circ \eta_1)$.

Recall (see McCord and Mischaikow (1992) for more details) that there exists a standard quotient map

$$\hat{\eta} : \hat{X} \to S^0 * S^1_1 * \ldots * S^1_{P-1} * \{P\}$$

where $*$ denotes the topological join of two spaces [1] and furthermore, there exists a homeomorphism

$$h : S^0 * S^1_1 * \ldots * S^1_{P-1} * \{P\} \to D^{2P-1}.$$

One now defines

$$Q := h \circ \hat{\eta} \circ \eta.$$

It is left to the reader to check that h can be chosen such that

$$\begin{aligned} Q(\pm 1, \phi, P, \tau, 0) &= (\pm 1, 0, \ldots, 0) \\ Q(\nu, \phi, P, 0, \ldots, \tau + p = 1, 0, \ldots, 0, p) &= z \quad \text{where } (z_{2p-1}, z_{2p}) = (\cos \phi_p, \sin \phi_p) \\ Q(\nu, \phi, P, \tau, P) &= (0, 0, \ldots, 0) \end{aligned}$$

As was mentioned above $\rho := Q \circ \tilde{\rho}$, so all that remains is to define $\tilde{\rho}$. The following proposition, which is a special case of Chapter 6.4 of Conley (1978) gives rise to $V : \mathcal{A} \to [0, P]$.

Proposition 7.1 *Let $\mathcal{A}$ be a compact invariant set with Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p = 0^{\pm}, 1, \ldots, P\}$. Then, there exists a continuous Lyapunov function $V : \mathcal{A} \to [0, P]$ such that*

1. *$M(p) \subset V^{-1}(p) \quad p = 0^{\pm}, 1, \ldots, P$,*
2. *$x \notin \cup_{p=0}^{P} M(p)$ implies that $V(x) > V(x \cdot t)$ for all $t > 0$.*

---

[1] Given two spaces A and B, their join is defined to be

$$A * B = \frac{A \times B \times [0,1]}{\sim}$$

where $(a, b, 0) \sim (a, b', 0)$ and $(a, b, 1) \sim (a', b, 1)$.

Let $\nu : \mathcal{A} \to S^0 = \{\pm 1\}$ be given by

$$\nu(x) = \begin{cases} 1 & \text{if } \omega(x) \subset M(0^+) \\ -1 & \text{if } \omega(x) \subset M(0^-) \\ 1 & \text{otherwise.} \end{cases}$$

Observe that ν is not continuous.

To define ϕ and τ special isolating neighborhoods for $M(p)$, $p = 1, \ldots, P-1$ need to be chosen. In particular, let $N_p \subset \text{int}W(p)$ be such an isolating neighborhood, i.e. it is assumed to have the property that given $x \in \mathcal{A}$ there exists a unique maximal interval $I_p(x) \subset \mathbf{R}$ such that

$$x \cdot \mathbf{R} \cap N_p = x \cdot I_p(x)$$

and

$$x \cdot \mathbf{R} \cap \partial N_p = x \cdot \partial I_p(x).$$

The reader is referred to McCord and Mischaikow (1992), Proposition 5.3 for a proof that such neighborhoods exist. It is convenient to represent

$$I_p(x) = [a_p(x), b_p(x)]$$

where $a_p(x), b_p(x) \in [-\infty, \infty]$, unless of course $I_p(x) = \emptyset$.

To define $\phi_p : \mathcal{A} \to S^1_p$, choose $s_0 \in S^1$ and let

$$\phi_p(x) = \begin{cases} \theta_p(x \cdot a_p(x)) & \text{if } a_p(x) \geq 0, \\ \theta_p(x) & \text{if } a_p(x) \leq 0 \leq b_p(x), \\ \theta_p(x \cdot b_p(x)) & \text{if } b_p(x) \leq 0, \\ s_0 & \text{if } I_p(x) = \emptyset. \end{cases}$$

Set

$$\phi(x) = (\phi_1(x), \ldots, \phi_{P-1}(x))\,.$$

Clearly, ϕ is not continuous since ϕ_p is not continuous. However, observe that the points of discontinuity of ϕ_p are contained in $\{x \in \mathcal{A} \mid a_p(x) = b_p(x)\}$.

Finally, to define τ, for $p = 1, \ldots P-1$, let

$$\lambda_p(x) = \begin{cases} \infty & \text{if } b_p(x) = \infty \text{ or } a_p(x) = -\infty \\ 0 & \text{if } I_p(x) = \emptyset \\ b_p(x) - a_p(x) & \text{otherwise.} \end{cases}$$

Set

$$\tau_p(x) = \frac{2}{\pi} \tan^{-1}(\lambda_p(x))$$

where $tan^{-1}(\pm\infty) = \pm\pi/2$. Define

$$\tau(x) = (\tau_1(x), \ldots, \tau_p(x)).$$

Proposition 7.2 *$\rho : \mathcal{A} \to D^{2P-1}$ is continuous.*

The proof of this proposition is omitted since it is almost identical to that of Proposition 6.4 of McCord and Mischaikow (1992).

It is now left to the reader to check that given an interval I

$$M(I) = \rho^{-1}(\Pi(I)). \tag{27}$$

7.2 ρ is surjective

The proof of Theorem 1.10 will be complete once it is shown that for every interval I

$$\rho(M(I)) = \Pi(I).$$

By equation (27), if $x \notin M(I)$ then $\rho(x) \notin \Pi(I)$. Therefore, it is sufficient to show that $\rho : \mathcal{A} \to D^{2P-1}$ is onto. To do this we decompose D^{2P-1} as follows. For $\lambda \in (0,1)$, let

$$X_\lambda = Q\left(\{(\nu, \phi, P, \tau, V) \in \tilde{X} \mid V = P - \lambda\}\right)$$

and observe that X_λ is homeomorphic to S^{2P-2}. Obviously, $X_\lambda \neq X_{\lambda'}$ if $\lambda \neq \lambda'$. Furthermore,

$$D^{2P-1} = 0 \cup \partial D^{2P-1} \cup \cup_{0<\lambda<1} X_\lambda.$$

Let

$$C_\lambda := \{x \in \mathcal{A} \mid V(x) = P - \lambda\} = \rho^{-1}(X_\lambda).$$

We shall show that $\rho : \mathcal{A} \to D^{2P-1}$ is onto by demonstrating that:

(a) $\rho(M(P)) = 0$

(b) $\rho(M(0^+, 0^-, 1, 2, \ldots P-1)) = \partial D^{2P-1}$

(c) $\rho(C_\lambda) = X_\lambda$ for each $\lambda \in (0,1)$.

Part (a) is trivial. Observe that

$$\partial D^{2P-1} = S^{2P-2} = \Pi(0^+, 0^-, 1, 2, \ldots, P-1).$$

Let $I = (0^+, 0^-, 1, 2, \ldots, P-1)$. It will be shown that

$$\left(\rho|_{M(I)}\right)^* : H^*(\Pi(I)) \to H^*(M(I))$$

and

$$(\rho|_{C_\lambda})^* : H^*(X_\lambda) \to H^*(C_\lambda)$$

are monomorphisms. Since $X_\lambda \approx \Pi(I) \approx S^{2P-2}$ this proves (b) and (c).

The proof that $\left(\rho|_{M(I)}\right)^*$ and $(\rho|_{C_\lambda})^*$ are monomorphism makes use of Floer's $*$-hyperbolicity discussed in Section 6 and the arguments of Section 7 of McCord and Mischaikow (1992). Observe that for $1 \leq p \leq P-1$, $\Pi(p)$ is normally hyperbolic and hence $*$-hyperbolic. Thus the same argument as in Section 6, which was used to prove (26), implies that there exists an isomorphism $\bar{D}_p : CH^{2p-1}(\Pi(p)) \to CH^{2p}(\Pi(p))$. Also by (26) there is an isomorphism $D_p : CH^{2p-1}(M(p)) \to CH^{2p}(M(p))$. Since

$$\begin{array}{ccc} CH^{2p-1}(M(p)) & \xrightarrow{D_p} & CH^{2p}(M(p)) \\ \downarrow \rho^{2p-1} & & \downarrow \rho^{2p} \\ CH^{2p-1}(\Pi(p)) & \xrightarrow{\bar{D}_p} & CH^{2p}(\Pi(p)) \end{array}$$

commutes and since D_p, $\bar{D}_p$ and ρ^{2p-1} are isomorphisms so is ρ^{2p}. This is the equivalent of Proposition 7.3 of McCord and Mischaikow (1992). The rest of the proof now follows verbatim from Section 7 of McCord and Mischaikow (1992).

8 Proof of Theorem 4.6

Set $s_{-1} = 0,\ s_P = 1$ and let

$$h_s(x) = \left([\frac{s_{k+1} - s}{s_{k+1} - s_k} d_1^{s_k} + \frac{s_k - s}{s_k - s_{k+1}} d_1^{s_{k+1}}] x_1^3, \ldots, [\frac{s_{k+1} - s}{s_{k+1} - s_k} d_n^{s_k} + \frac{s_k - s}{s_k - s_{k+1}} d_n^{s_{k+1}}] x_n^3 \right) \quad (28)$$

for $s_k \leq s \leq s_{k+1},\ k = 0, 1, \ldots, P-1$.

Note , that for $s = s_k$ we have

$$h_{s_k}(x) = (d_1^{s_k} x_1^3, \ldots, d_n^{s_k} x_n^3)$$

and for $s = s_{k+1}$ we have

$$h_{s_{k+1}}(x) = (d_1^{s_{k+1}} x_1^3, \ldots, d_n^{s_{k+1}} x_n^3).$$

We will show that $d_j^{s_k},\ k = 0, \ldots, P-1$ can be assigned in a way that if at s_k a Hopf bifurcation takes place, then it is a supercritical Hopf bifurcation. For the purpose of this proof we restrict ourself to the case $\Delta = -1,\ n-$odd and $J < n$ to make explicit calculations possible. In this case all bifurcations at $s_k,\ k = 1, \ldots, P-1$ are the Hopf bifurcations.

We use the averaging techniques of Chow and Mallet-Paret (1977). Straightforward computation shows that the linear part of $F(s_k, x)$ $k = 0, \ldots, P-1$ is

$$L(\cos \frac{(k+1)\pi}{n})$$

and that the bifurcating pair of eigenvalues is

$$u_{1,2}^k = \pm i \sin \frac{(k+1)\pi}{n}.$$

Lemma 8.1 *Eigenvectors w_1^k, w_2^k corresponding to $u_{1,2}^k$ are*

$$w_1^k = (1, -\cos \frac{n-1}{n}(k+1)\pi, \ldots, -\cos \frac{1}{n}(k+1)\pi)^T$$

$$w_2^k = (0, -\sin \frac{n-1}{n}(k+1)\pi, \ldots, -\sin \frac{1}{n}(k+1)\pi)^T.$$

Proof. If we choose the first entry in the (complex) eigenvector to be 1, then straightforward computation leads to w_1^k, w_2^k. □

Let us fix some $k,\ k = 0, 1, \ldots, P-1$. We will drop the index k in the sequel.

Let us consider new coordinates $z = (w_1, w_2, y_1, \ldots, y_{n-2}) \in R^n$,where $y_1, \ldots, y_{n-2}$ are perpendicular to w_1 and w_2 and $span\{y_1, \ldots, y_{n-2}\} \oplus span\{w_1, w_2\} = R^n$.

We will denote $w = (w_1, w_2)$ and $y = (y_1, \ldots, y_{n-2})$. Following Chow and Mallet-Paret (1977) we can write our system in this coordinates as follows:

$$\begin{aligned} \dot{w} &= \alpha E(\alpha) y + F(y, \alpha) y^2 + A_p(\alpha) w + G(y, \alpha) wy + \\ &\quad + B_2(y, \alpha) w^2 + B_3(y, \alpha) w^3 + \ldots \qquad (29) \\ \dot{y} &= \alpha H(\alpha) w + J(w, \alpha) w^2 + A_Q y + \alpha M(\alpha) y + N(w, \alpha) wy + \\ &\quad + \Gamma_2(w, \alpha) y^2 + \Gamma_3(w, \alpha) y^3 + \ldots \qquad (30) \end{aligned}$$

Changing to the polar coordinates $w = (r\cos\theta, r\sin\theta)$ we get:

$$\begin{aligned}\dot{r} &= [\alpha E_1(\theta\alpha)y + F_1(\theta,y,\alpha)y^2] + r[\alpha + G_2(\theta,y,\alpha)y] + \\ &\quad + r^2C_3(\theta,y,\alpha) + r^3C_4(\theta,y,\alpha) + \dots \qquad (31)\\ \dot{\theta} &= \frac{1}{r}[\alpha E_1^*(\theta\alpha)y + F_1^*(\theta,y,\alpha)y^2] + [\omega(\alpha) + G_2^*(\theta,y,\alpha)y] + \\ &\quad + rD_3(\theta,y,\alpha) + r^2D_4(\theta,y,\alpha) + \dots \qquad (32)\\ \dot{y} &= \text{ as above but with } w = (r\cos\theta, r\sin\theta) \qquad (33)\end{aligned}$$

Here $\alpha := s - s_k,\ s \in (s_k - \delta, s_k + \delta)$ is the bifurcation parameter close to zero and subscripts in $C_2, C_3, \dots$ indicate that the given function f_j is homogenuous of degree j in the space variable.

Also

$$C_j(\theta,y,\alpha) = \cos\theta B^1_{j-1}(\cos\theta,\sin\theta,y,\alpha) + \sin\theta B^2_{j-1}(\cos\theta,\sin\theta,y,\alpha)$$

$$D_j(\theta,y,\alpha) = \cos\theta B^2_{j-1}(\cos\theta,\sin\theta,y,\alpha) - \sin\theta B^1_{j-1}(\cos\theta,\sin\theta,y,\alpha)$$

Scaling $r \longrightarrow \epsilon r,\ y \longrightarrow \epsilon y, \alpha \longrightarrow \epsilon\alpha$ we get

$$\begin{aligned}\dot{r} &= \epsilon[\alpha r + r^2C_3(\theta,\epsilon y,\epsilon\alpha) + F_1(\theta,\epsilon y,\epsilon\alpha)y^2 + rG_2(\theta,\epsilon y,\epsilon\alpha)y] \\ &\quad + \epsilon^2r^3C_4(\theta,\epsilon y,\epsilon\alpha) + O(\epsilon^3)\\ \dot{\theta} &= \omega_0 + \epsilon[\alpha\omega'(0) + rD_3(\theta,\epsilon y,\epsilon\alpha) + \frac{\alpha}{r}E_1^*(\theta,\epsilon\alpha)y + \\ &\quad + \frac{1}{r}F_1^*(\theta,\epsilon y,\epsilon\alpha)y^2 + G_2^*(\theta,\epsilon y,\epsilon\alpha)y] + O(\epsilon^2) \qquad (34)\\ \dot{y} &= A_Qy + \epsilon[\alpha H(\alpha)w + J(\epsilon w,\epsilon\alpha)w^2 + \alpha M(\epsilon\alpha)y + \\ &\quad + N(\epsilon w,\epsilon\alpha) + \Gamma_2(\epsilon w,\epsilon\alpha)y^2] + O(\epsilon^2)\end{aligned}$$

Theorem 8.2 [Chow and Mallet-Paret (1977), Theorem 5.1] *Given system (34) there exists a coordinate change*

$$\bar{r} = r + \epsilon u_1(r,\theta,\alpha,\epsilon) + \epsilon z(r,\theta,\alpha,\epsilon)y + \epsilon^2u_2(r,\theta,\alpha,\epsilon)$$

transforming (34) into the averaged system of the form

$$\begin{aligned}\dot{\bar{r}} &= \epsilon\alpha\bar{r} + \epsilon^2\bar{r}^3K + O(\epsilon|y|^2) + O(\epsilon^2|y|) + O(\epsilon^3)\\ \dot{\theta} &= \omega_0 + O(\epsilon) \qquad (35)\\ \dot{y} &= A_Qy + O(\epsilon)\end{aligned}$$

where K is the constant

$$\begin{aligned}K &= K^* + K^{**}\\ K^* &= \frac{1}{2\pi}\int_0^{2\pi} C_4(\theta,0,0) - \frac{1}{\omega_0}C_3(\theta,0,0)D_3(\theta,0,0)d\theta \qquad (36)\\ K^{**} &= \frac{1}{2\pi}\int_0^{2\pi} w^*(\theta)J(0,0)(cos\theta, sin\theta)^2d\theta,\end{aligned}$$

where w^* is the unique 2π-periodic solution of

$$G_2(\theta,0,0) + w^*(\theta)\omega_0 + w^*(\theta)A_Q = 0.$$

Recall that for each (w, α), $J(w, \alpha)$ is a bilinear form in the w-space R^2 taking values in the y-space; in the theorem $J(0,0)$ acts on the point $(\cos\theta, \sin\theta) \in R^2$.

Theorem 8.3 [Chow and Mallet-Paret (1977), Theorem 6.2] *Given (35) then in the original unaveraged , unscaled system there is a unique periodic solution bifurcating from the origin either for $\alpha > 0$ if $K < 0$ or for $\alpha < 0$ if $K > 0$.*

In order to prove Theorem 4.6 we need to show that we can choose $d_j^{s_k}$ in such a way that $K < 0$. Now we compute the constant K for $s = s_k$.

Let us rewrite $\dot{x} = F(s_k, x)$ as

$$\dot{x} = Ax + f(x)$$

where $A = L(\cos\frac{(k+1)\pi}{n})$ and $f = f_{s_k}$.

Let U be a matrix with $w_1, w_2, y_1, \ldots, y_{n-2}$ as columns. Then

$$\dot{z} = U^{-1}AUz + U^{-1}f(Uz) \tag{37}$$

where $z = (w_1, w_2, y_1, \ldots, y_{n-2})^T$.

Now

$$f(Uz) = (d_1(w_{11}w_1 + w_{12}w_2 + uy)^3, d_2(w_{21}w_1 + w_{22}w_2 + vy)^3, \ldots)^T$$

where w_{ij} is the i-th component of the vector w_j and u, v are n-2–vectors. Also

$$U^{-1}f(Uz) = \begin{pmatrix} d_1p_{11}(w_{11}w_1 + w_{12}w_2 + uy)^3 + d_2p_{12}(w_{21}w_1 + w_{22}w_2 + vy)^3 + \ldots \\ d_1p_{21}(w_{11}w_1 + w_{12}w_2 + uy)^3 + d_2p_{22}(w_{21}w_1 + w_{22}w_2 + vy)^3 + \ldots \\ \vdots \end{pmatrix}$$

where p_{ij} is the entry in the i-th row and j-th column of the matrix U^{-1}.

Turning now to the polar coordinates in the w_1, w_2-plane; $w_1 = r\cos\theta$, $w_2 = r\sin\theta$, by a straightforward, but tedious computation one can show that

$$C_3(\theta, 0, 0) = 0, \quad J(0,0) = 0$$

and

$$\begin{aligned} C_4(\theta,0,0) &= \cos^4\theta[d_1p_{11}w_{11}^3 + d_2p_{12}w_{21}^3] + \\ &+cos^3\theta\sin\theta[d_1p_{21}w_{11}^3 + d_2p_{22}w_{21}^3 + 3d_1p_{11}w_{11}^2w_{12} + 3d_2p_{12}w_{21}^2w_{22}] + \\ &+cos^2\theta\sin^2\theta[3d_1p_{11}w_{11}w_{12}^2 + 3d_2p_{12}w_{21}w_{22}^2 + 3d_1p_{21}w_{11}^2w_{12} + 3d_2p_{22}w_{21}^2w_{22}] + \\ &+cos\theta\sin^3\theta[d_1p_{11}w_{12}^3 + d_2p_{12}w_{22}^3 + 3d_1p_{21}w_{11}^2w_{12} + 3d_2p_{22}w_{21}^2w_{22}] + \\ &+\sin^4\theta[d_1p_{21}w_{12}^3 + d_2p_{22}w_{22}^3] \end{aligned} \tag{38}$$

Intergarting (36) with (38) we have

$$\begin{aligned} K &= \frac{1}{2\pi}\int_0^{2\pi} C_4(\theta,0,0)d\theta = \\ &= \frac{3}{8}[d_1(p_{11}w_{11} + p_{21}w_{12})(w_{11}^2 + w_{12}^2) + d_2(p_{12}w_{21} + p_{22}w_{22})(w_{21}^2 + w_{22}^2)]. \end{aligned} \tag{39}$$

Now we use Lemma 8.1 to get

$$K = \frac{3}{8}[d_1 p_{11} - d_2(p_{12}\cos\frac{n-1}{n}(k+1)\pi + p_{22}\sin\frac{n-1}{n}(k+1)\pi)].$$

We claim that $p_{11} \neq 0$. Assume to the contrary that $p_{11} = 0$. Then by definition of p_{ij} we have

$$e_1 = p_{11}w_1 + p_{21}w_2 + p\cdot y = p_{21}w_2 + py.$$

If $p_{21} \neq 0$ then $w_2 = \frac{1}{p_{21}}(e_1 - py)$ and

$$0 \neq ||w_2|| =< w_2, w_2 >= \frac{1}{p_{21}}(< e_1, w_2 > - < py, w_2 >) = 0$$

since $w_{12} = 0$, which is a contradiction.

Therefore $p_{21} = 0$. But then $e_1 = py$ and thus $e_1 \perp span\{w_1, w_2\}$. This, however, contradicts the fact that $< e_1, w_1 >= w_{11} = 1$.

Hence we conclude that $p_{11} \neq 0$ and since we can choose d_1, d_2 arbitrary we can make $K < 0$. Since k was arbitrary, this proves the Theorem 4.6. □

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