Attractors in continuous-time switching networks

Tomáš Gedeon,*†
Department of Mathematical Sciences
Montana State University
Bozeman, MT 59717-0240, USA

January 24, 2003

Abstract

We consider a system of equations with discontinuous right hand side, which arise as models of gene and neural networks. We study attractors in $\mathbf{R}^4$ which lie in a set of orthants in the form of figure eight. We find that if the attractor is symmetric with respect to these two loops, then the only possible attractor is a periodic orbit which traverses both loops once.

We show that without the symmetry the set of admissible attractors include periodic orbits which follow one loop k times and other loop once, for any k. However, we also show that no trajectory in an attractor can traverse both loops more then once in a row.

Keywords: Attractors, continuous-time switching networks, additive neural networks.

1 Introduction

In this paper we study dynamics of the equations, introduced by L. Glass [5]

$$\dot{x}_i = -x_i + \Lambda_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \tag{1}$$

The functions Λ_i depend only on the signs of the variables $x_1, \ldots, x_n$ and hence are constant on the interior of every orthant in $\mathbf{R}^n$. This system is a generalization of a *infinite gain additive neural network* which we describe next. It was also used as a model for gene networks, where the effect of each gene on another one in the network only depends on two states of a gene - an ON state and an OFF state. This simple model may lead to equation of the type (1). For more details and more applications one may consult Mestl et.al [11] and Edwards [1].

The additive neural network is the set of equations of the form

$$\dot{x}_i = -x_i + \sum_{j \neq i} w_{ij} f_j(x_j), \quad i = 1, \ldots, n. \tag{2}$$

where w_{ij} are interpreted as the synaptic connection weights and functions $f_j(x_j)$ are nonlinear sigmoidal gating functions with $f_j(0) = 0$. The study of this model goes back to Grossberg [7, 8] and Hopfield [10]. The *gain* of f_j is the derivative $f_j'(0)$. The correspondence between stable equilibria of (2) for symmetric matrix of weights $W = [w_{ij}]$ and the stable fixed points of asynchronous content addressable memory as the gain approaches infinity was studied in Hopfield [10]. His argument was corrected and completed by T. Troyer [13].

The natural way to take the limit of the gain to infinity is to consider (2) with discontinuous nonlinearities

$$f_j(x_j) = \begin{cases} u_j & \text{for } x_j > 0 \\ -v_j & \text{for } x_j < 0 \end{cases}. \tag{3}$$

*AMS subject classification: 34C25, 34D45, 92B20
†This research partially supported by DMS-NSF grant No. 291222

It is easy to see that such a system is of the form (1); however, the class (1) is broader.

In the interior of any orthant (1) can be solved explicitly. Indeed, in a fixed orthant $\mathcal{O}$ in $\mathbf{R}^n$ the value of Λ_i is constant, $p_i := \Lambda_i(\mathcal{O})$. Thus we can integrate equations (1) in orthant $\mathcal{O}$ to get

$$x_i(t) = p_i + (x_i(0) - p_i)e^{-t}. \tag{4}$$

The point $p(\mathcal{O}) := (p_1, \ldots, p_n)$ is called the *target point* of the orthant $\mathcal{O}$. Thus the dynamics in every orthant is linear and focused toward the target point. Once the solution hits the boundary of $\mathcal{O}$, the target point changes. For this reason the target point was also called *ghost attractor* by Troyer [13].

We assume that

$$\text{the value of } \Lambda_i(x_1, \ldots, x_n) \text{ does not depend on } x_i \qquad (\mathbf{H1}).$$

To avoid unnecessary technical problems we assume generic condition that

$$\text{every target point } p \text{ lies in the interior of an orthant} \qquad (\mathbf{H2}).$$

Observe that the assumption (**H1**) guarantees that the trajectories of (1) can be continued from orthant $\mathcal{O}_A$ to the next orthant $\mathcal{O}_B$, if the trajectory hits a codimension one hyperplane separating $\mathcal{O}_A$ and $\mathcal{O}_B$. We define the vector field on such a hyperplane to be the closure of the vector field in $\mathcal{O}_B$. We denote this C^0, piecewise smooth flow by $\Phi(x,t)$. Observe that a following weaker version of assumption (**H1**) would suffice to draw the same conclusion.

$$\text{The value of } \Lambda_i(x_1, \ldots, x_n) \text{ does not depend on the sign of } x_i \qquad (\mathbf{H1'}).$$

We will see that the difference between (**H1**) and (**H1'**) has a major impact on a structure of certain attractors in $\mathbf{R}^4$.

If the trajectory hits any codimension p hyperplane, $p \geq 2$,

$$H := \{x \in \mathbf{R}^n \mid x_{i_1} = \ldots = x_{i_p} = 0\},$$

then the flow is not well defined, since this hyperplane is in the closure of more then two orthants. In this case a more general notion of *differential inclusions* [3] is appropriate. We deal with this problem by restricting the system (1) to a domain

$$D := \{x \in \mathbf{R}^n \mid \Phi(x,t) \notin H \quad \text{for all } t \text{ and } H\}.$$

An important tool in analyzing the dynamics of (1) will be the *graph of the dynamics* G. Every orthant in $\mathbf{R}^n$ will be represented as a vertex of the graph and a boundary between two orthants as an edge. We assign a direction to the edge which is consistent with the flow direction across the corresponding hyperplane. The assumption **(H1)** implies that each edge has a unique orientation and thus the graph G is well defined.

One of the reasons to study (1) is the possibility that we can reduce questions about the dynamics of system (1) with infinite gain to questions about the structure of the (finite) graph G. There are some positive answers in this direction. We say that $S \in G$ is an *attracting* set of vertices, if all edges with one end in S are oriented toward S. The corresponding set of orthants, $\mathcal{O}_S$, is an *attracting set of orthants*. If the attracting set of orthants consist of a single orthant $\mathcal{O}_V$, then the target point of this orthant lies in $\mathcal{O}_V$. This target point is attracting all points in $\mathcal{O}_V$ and hence is a local attractor in (1).

An *N-dimensional cyclic attractor* C in G is an attracting cycle in G, which is not contained in any lower dimensional sub-cube of G.

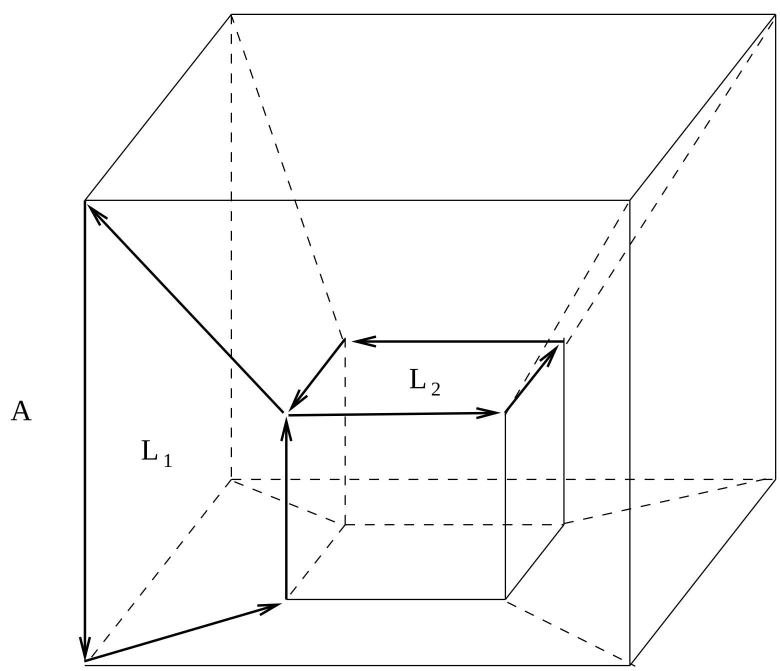
Theorem 1.1 ([6]) *Given an N-dimensional cyclic attractor C in G, the corresponding set $\mathcal{O}_C$ either admits a unique stable periodic orbit, or all solutions in $\mathcal{O}_C$ converge to the origin.*

These two results raise some hope that the graph G can be used successfully to study dynamics of (1). This hope was further enhanced by construction of a Morse decomposition of the invariant set S of the related *ray flow* (R-flow) based on the graph G. The construction of the R-flow is based on an observation by Mestl et. al. [11], that any two points on the same ray through the origin eventually converge to each other. The R-flow tracks the time evolution of rays. We recall the construction of R-flow in section 2. For details one may consult Gedeon [4].

The existence of the ray flow in (1) precludes complicated invariant sets in $\mathbf{R}^3$, since the R-flow dynamics takes place effectively on two-dimensional sphere S^2. This was recognized by Mestl et. al [11] when they constructed a chaotic invariant set of (1) in $\mathbf{R}^4$. They explored a nice feature of system (1); it is easy to compute all dynamically relevant objects (Poincaré maps, stable and unstable manifolds) explicitly for a fixed system (1). They searched parameter space on computer until they found parameters and functions Λ_i where dynamics looked chaotic and then they proved that this indeed is the case. They showed that there is a transverse homoclinic point for a certain Poincaré map and they constructed a trapping region for what seems to be numerically a chaotic attractor. However, the precise relationship between the chaotic attractor and the transverse homoclinic orbit is not known. More recently, Edwards [2] has proven the existence of a system (1) in $\mathbf{R}^4$ with an aperiodic attractor.

In this paper we take a different approach. We start with an attracting set of orthants A in $\mathbf{R}^4$ in the form of figure eight (see Figure 1). We investigate how the invariant set in A changes as we change the parameters of the problem. This is a complicated problem since there are seven target points associated with the set A and hence there are a priori 28 parameters. We discover, however, that there are only six true parameters and in fact a lot of dynamics only depends on a certain combinations of these parameters.

Figure 1: The graph G with attracting set of orthants A in the form of figure eight. All unmarked edges point toward A.



We present now main results. Given an orthant $\mathcal{O}$ the boundary components will be called *walls* of an orthant. Each wall corresponds to an edge of the graph G. They are divided into *incoming* and *outgoing* walls of $\mathcal{O}$, depending on the direction of the flow. An orthant is called *splitting orthant* if it has at least two outgoing walls. We denote by π the union of incoming walls into the splitting orthant of the set A in figure 1. Let (a_1, b_1, c_1, d_1) be the target point of the splitting orthant. We assume that

$$a_1 = c_1, \qquad b_1 = d_1. \qquad\qquad (\mathbf{H3}).$$

This is a technical assumption which makes the construction of the Poincaré map easier. We believe that it can be dropped without affecting our results below, but for a steep price in increase of technical difficulties.

The set π will serve as a Poincaré section of the flow in A. Set A consists of two loops, L_1 and L_2. The Poincaré map consists of two maps M_1 and M_2, which represent the flow through L_1 and L_2 respectively.

We associate to every $x \in \pi$ a sequence of numbers $z(x) := (a_1, a_2, \ldots)$ such that

$$a_i = j \quad \text{if } i\text{-th iterate of } x \text{ is in the domain of } M_j.$$

The sequence $z(x)$ is biinfinite, if and only if, x is in the invariant set $\mathrm{Inv}(A)$ of A. Such a sequence will be called a*pattern of* x. As an example a fixed point of M_1 will have a pattern $(1, 1, \ldots)$ and a periodic point under $M_2 \circ M_1$ will have a pattern $(1, 2, 1, 2, \ldots)$. Every periodic orbit in $\mathrm{Inv}(A)$ has a periodic pattern $z(x)$. We say that a collection $(z_1, z_2, \ldots, z_n)$ is *the type of the periodic orbit*, if the periodic pattern $z(x)$ of the periodic orbit has z_1 symbols of one type, followed by z_2 symbols of the other type, followed by z_3 symbols of the first type and so on. The types of the above patterns are $(1, 0)$ and $(1, 1)$, respectively.

Theorem 1.2 *Consider (1) in* $\mathbf{R}^4$ *with the attracting set* A. *Assume* **(H1)**,**(H2)** *and* **(H3)**. *For any* $k \geq 1$ *there is exist a choice of* Λ_i *such that the system (1) has a unique attracting periodic orbit of type* $(k, 1)$. *The same is true for the type* $(1, k)$.

In Theorem 3.1 we explicitly show how to find the appropriate Λ_i for a given k. Observe that since constructed periodic orbits are attracting, the same will be true for all nearby functions Λ_i. This result shatters the idea that the graph G provides any information about the structure of the invariant sets.

There are some restrictions on possible attractors in A.

Theorem 1.3 *Consider (1) in* $\mathbf{R}^4$ *with the attracting set* A. *Assume* **(H1)**,**(H2)** *and* **(H3)**. *Let* $z(x)$ *be a pattern of* $x \in \mathrm{Inv}(A)$. *Then either there are no two symbols* 1 *in a row in* $z(x)$ *or there are no two symbols* 2 *in a row in* $z(x)$.

In addition, we can show that there is no chaotic attractor in A, if the maps M_1 and M_2 are symmetric. Notice that along loop L_1 the variables x_1 and x_2 do not change sign while along loop L_2 the variables x_3 and x_4 do not change sign. This suggests that M_1 and M_2 can be related by the change of variables $(x, y, z, w) \to (z, w, x, y)$.

Theorem 1.4 *Consider (1) in* $\mathbf{R}^4$ *with the attracting set* A. *Assume* **(H1)**,**(H2)** *and* **(H3)** *and, in addition, that the maps* M_1 *and* M_2 *are related by the change of variables* $(x, y, z, w) \to (z, w, x, y)$.

Then the invariant set which attracts all points in the interior of A *is a pair of fixed points on the boundary of* A, *a unique periodic orbit in* A *of type* $(1, 1)$, *or the origin.*

Theorem 1.4 is a surprising generalization of the Theorem 1.1. If assumption (**H1**) is relaxed to assumption (**H1'**) the Theorem 1.4 is not true. There is a choice of functions Λ_i for which all periodic orbits of types $(k, 1)$ and $(1, s)$ coexist for all k and s. They are only marginally stable, however, since there is a one-dimensional family of each of these orbits.

2 Preliminaries

2.1 The R-flow

As we saw in the introduction, the trajectories of (1) are straight lines inside every orthant. We may compute a transition function from an incoming wall W_1 to an outgoing wall W_2 through an orthant $\mathcal{O}$. This function takes an initial value $\mathbf{x}$ on an W_1 and associates to it the intersection $\mathbf{y}$ of the solution with W_2. Assume that on the wall W_2 we have $x_j = 0$ and that the target point of the orthant $\mathcal{O}$ is $p = (p_1, \ldots, p_n)$. Solving the j-th equation for the time of transition t^* and then substituting to the other equations we find that

$$y_i = \frac{x_i - (p_i/p_j)x_j}{1 - x_j/p_j}. \tag{5}$$

In the vector notation (see [6] and [11])

$$\mathbf{y} = \mathcal{M}_{\mathcal{O}}(\mathbf{x}) = \frac{M\mathbf{x}}{1 + \mathbf{c}^t\mathbf{x}}, \tag{6}$$

where $M \in \mathbf{R}^{n\times n}$ and the transposed vector $\mathbf{c}$ has zero entries except $\mathbf{c}_j = -1/p_j$. By assumption (**H2**) we have that $\mathbf{c}^t\mathbf{x} > 0$. Observe that the formula (5) does not depend on the wall W_1 in other way than initial condition. So this formula is valid for the transition from any incoming wall to the outgoing wall W_2.

The map $\mathcal{M}_{\mathcal{O}}$ is a linear fractional transformation and a composition of two such transformations is again a linear fractional transformation of the same form..

Consider two points, $x_1 = k_1 v$ and $x_2 = k_2 v$, on a ray starting at the origin. The linearity of the flow in $\mathcal{O}$ guarantees that the trajectories starting at x_1 and x_2 are collinear. By this we mean the following: given a point $z = \Phi(x_1, t)$, $0 \leq t \leq t^*(x_1)$ where $t^*(x_1)$ is given by $y_1 = \Phi(x_1, t^*(x_1))$, there is a point $w = \Phi(x_2, \bar{t})$, $0 \leq \bar{t} \leq t^*(x_2)$, such that z and w are collinear. Notice, that $t^*(x_1) \neq t^*(x_2)$ in general. However, the map from an incoming wall W_1 to an outgoing wall W_2 in an orthant $\mathcal{O}$ maps the ray through a point x to the ray through a point Mx, see (6). Since the flow $\Phi(x,t)$ is linear in $\mathcal{O}$, the flow lines of $\Phi(x,t)$ starting at the ray through x span the plane containing rays through x and Mx. We rescale time in flow $\Phi(x,t)$ so that the points initially on a ray stay on a ray throughout the interior of the orthant $\mathcal{O}$. The rescaling can be made continuous by the continuous dependence on initial conditions. After rescaling we project the flow $\Phi(x,t)$ onto the unit sphere to obtain the *ray flow (R-flow)* $\varphi(x,t)$ on a subset $\mathcal{D}$ of S^{n-1}. The set $\mathcal{D}$ is defined as

$$\mathcal{D} := \{z \in S^{n-1} \mid [\varphi(z,t)]_i = 0 \text{ for at most one } i \in \{1,\ldots,n\}\},$$

where $[\varphi(z,t)]_i$ is the i-th component of the vector $\varphi(z,t)$. The walls W in $\mathbf{R}^n$ divide the sphere S^{n-1} into regions, each of which lies in one orthant. Every $x \in \mathcal{D}$ corresponds to a ray in $\mathbf{R}^n$ along the vector x.

Most of the time we will work with the maps mapping the rays in incoming walls to the rays on the outgoing walls, rather then directly with R-flow. We again emphasize that these maps are linear

$$\mathbf{M}(x) = M\mathbf{x}. \tag{7}$$

Notice that x on the left hand side is a point in $\mathcal{D}$ while $\mathbf{x}$ on the right hand side is a vector in $\mathbf{R}^n$ with the same components as x. Observe that a fixed point of $\mathbf{M}$ corresponds to an eigenvector of the matrix M. An eigenvector of M with a real eigenvalue larger then one corresponds to a locally attracting fixed point of $\mathbf{M}$, while an eigenvector with a real eigenvalue less then one corresponds to a repelling fixed point of $\mathbf{M}$. We shall denote the vector on the right hand side by the sams symbol as the point it corresponds to in $\mathcal{D}$.

2.2 From R-flow to real flow

In this section we recall result of Gedeon [4], which allows to decide which invariant set of the R-flow correspond to an invariant set of the flow of (1). Consider a set of orthants $\mathcal{O}_S$ and the corresponding subgraph $G(S)$. Fix a vertex A from the graph $G(S)$ and assume that $i \in w_{in}(A)$ and $j \in w_{out}(A)$. Let $S := Inv(\mathcal{O}_S \cap \mathcal{D})$ be the invariant set in $\mathcal{O}_S$.

Definition 2.1 We say that the transition $i \to j$ at vertex A is an *expanding transition* if for every $x \in S \cap \{x_i = 0\}$.

$$\sum_{k\neq i,j} p_k x_k > 0, \qquad p_j^2 < \sum_{k\neq j} p_k^2 \tag{8}$$

where $p = (p_1, \ldots, p_n)$ is the target point for vertex A.

Theorem 2.2 *If all transitions in $G(S)$ are expanding then the invariant set S under R-flow is homeomorphic to an invariant set S' of the system (1).*

2.3 The index of a vector field

In this subsection we recall some basic properties of an index of a vector field. Our primary reference is Milnor [12]. Consider an open set $U \subset \mathbf{R}^n$ and a smooth vector field

$$v : U \to \mathbf{R}^n$$

with an isolated zero at the point $z \in U$. The function $\bar{v}(x) = \frac{v(x)}{||v(x)||}$ maps a small sphere centered at z into the unit sphere. After rescaling $\bar{v} : S^{n-1} \to S^{n-1}$. The degree of this mapping is called *index I_z of v* at the zero z. Let $D^n \subset U$ be a disc. Then on can define index I_D of v on the boundary of D as the degree of $\bar{v} : \partial D^n \to S^{n-1}$. It is clear that

$$I_D = \sum_{z \in D} I_z.$$

It will be useful to compute the local index I_z at an isolated zero z using the derivative dv_z of v. We think of a vector field v as a smooth map from $U \to \mathbf{R}^n$ so the derivative makes sense. The vector field v is non-degenerate at z if the linear transformation dv_z is nonsingular.

Lemma 2.3 (Lemma 6.4,[12]) *The index I_z of v in a non-degenerate zero z is either $+1$ or -1 according as the determinant of dv_z is positive or negative.*

2.4 Double loop invariant set

The goal of this section is to develop some tools to study a set of equations of the type (1) whose graph of interactions has an attractive set of orthants, A, in a form of figure eight (Figure 1). Let S' be an invariant set in A which attracts the interior of the set A. We shall investigate the corresponding invariant set S under the R-flow on $S^3 \cap A$.

We denote each orthant in $\mathbf{R}^4$ by its *signature* . Thus $(+,-,-,+)$ denotes an orthant $\{x \mid x_1 > 0, x_2 < 0, x_3 < 0, x_4 > 0\}$. Similarly, each wall between the orthants has a signature with one zero in it; $(+,-,0,+)$ denotes the wall $\{x \mid x_1 > 0, x_2 < 0, x_3 = 0, x_4 > 0\}$. With this we denote the two loops in G on Figure 1 by

$$L_1 : (+,-,+,-) \to (+,-,+,+) \to (+,-,-,+) \to (+,-,-,-) \to (+,-,+,-)$$
$$L_2 : (+,-,+,-) \to (+,+,+,-) \to (-,+,+,-) \to (-,-,+,-) \to (+,-,+,-).$$

Let Π_1 be a wall $(+,-,0,-)$ and Π_2 be the wall $(0,-,+,-)$. Then the union of $\Pi_1 \cup \Pi_2$ is a Poincaré section of the invariant set in A. The Poincaré map is a collection of two linear fractional transformations (6) $\mathcal{M}_i : \Pi_1 \cup \Pi_2 \xrightarrow{L_i} \Pi_i$ where $i = 1, 2$. In order to study the Poincaré map for the R-flow, we choose a particular representation of the phase space $\mathcal{D} \cap S^3$. We choose as our representation of S^3 a three dimensional simplicial complex C with sixteen vertices of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$. The intersection of a wall and the complex C is a 2-dimensional simplex. The intersection of the wall Π_1 with C, denoted by P_1, is the simplex with vertices $U_1 := (1,0,0,0)$, $U_2 := (0,-1,0,0)$ and $U_4 := (0,0,0,-1)$. The intersection of the wall Π_2 with C, denoted by P_2, is a 2-simplex with vertices $U_2, U_3 := (0,0,0,1)$ and U_4. Let $P := P_1 \cup P_2$. Thus the Poincaré map for the R-flow consists of two maps of the form

$$\mathbf{M}_i : P_1 \cup P_2 \xrightarrow{L_i} P_i, \qquad i = 1, 2.$$

It is easy to see that $\mathbf{M}_1(U_i) = U_i$ for $i = 1, 2$ and $\mathbf{M}_2(U_j) = U_j$ for $j = 3, 4$.

Let us denote the target points along the loop L_1 (in order)

$$\begin{aligned} T_1 &= (a_1, b_1, c_1, d_1) \in (+,+,+,+), & T_2 &= (a_2, b_2, c_2, d_2) \in (+,-,-,+), \\ T_3 &= (a_3, b_3, c_3, d_3) \in (+,-,-,-), & T_4 &= (a_4, b_4, c_4, d_4) \in (+,-,+,-). \end{aligned} \tag{9}$$

In order to understand the return maps $\mathbf{M}_i$ we shall study the corresponding matrices M_i, $i = 1, 2$, (see (7)).

The matrix M_1 is a composition $M_1 = A_4 \circ A_3 \circ A_2 \circ A_1$ where the matrices A_i are

$$A_i = \begin{bmatrix} 1 & 0 & 0 & -a_i/d_i \\ 0 & 1 & 0 & -b_i/d_i \\ 0 & 0 & 1 & -c_i/d_i \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_j = \begin{bmatrix} 1 & 0 & -a_j/c_j & 0 \\ 0 & 1 & -b_j/c_j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -d_j/c_j & 1 \end{bmatrix},$$

for $i = 1, 3$ and $j = 2, 4$. The composition M_1 has the form

$$M_1 = \begin{bmatrix} 1 & 0 & D_1 & D_2 \\ 0 & 1 & D_3 & D_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_1 & E_2 \end{bmatrix}, \tag{10}$$

where the entries

$$\begin{aligned} D_1 &= -\frac{a_2}{c_2} + \frac{d_2}{c_2}\frac{a_3}{c_3} - \frac{a_4}{c_4}\frac{c_3}{d_3}\frac{d_2}{c_2}, \; D_2 = -\frac{a_1}{d_1} + \frac{a_2}{c_2}\frac{c_1}{d_1} - \frac{c_1}{d_1}\frac{d_2}{c_2}\frac{a_3}{d_3} + \frac{c_1}{d_1}\frac{d_2}{c_2}\frac{c_3}{d_3}\frac{a_4}{c_4} \\ D_3 &= -\frac{b_2}{c_2} + \frac{d_2}{c_2}\frac{b_3}{c_3} - \frac{b_4}{c_4}\frac{c_3}{d_3}\frac{d_2}{c_2}, \; D_4 = -\frac{b_1}{d_1} + \frac{b_2}{c_2}\frac{c_1}{d_1} - \frac{c_1}{d_1}\frac{d_2}{c_2}\frac{b_3}{d_3} + \frac{c_1}{d_1}\frac{d_2}{c_2}\frac{c_3}{d_3}\frac{b_4}{c_4} \\ E_1 &= -\frac{d_4}{c_4}\frac{c_3}{d_3}\frac{d_2}{c_2}, \qquad E_2 = \frac{c_1}{d_1}\frac{d_4}{c_4}\frac{c_3}{d_3}\frac{d_2}{c_2}. \end{aligned} \tag{11}$$

Similarly, we denote the target points along the loop L_2 (in order) $(a_1, b_1, c_1, d_1) \in (+,+,+,+)$, $T_5 = (a_5, b_5, c_5, d_5) \in (-,+,+,-)$, $T_6 = (a_6, b_6, c_6, d_6) \in (-,-,+,-)$ and $T_7 = (a_7, b_7, c_7, d_7) \in (+,-,+,-)$. Note, that (a_7, b_7, c_7, d_7) and (a_4, b_4, c_4, d_4) are in the same orthant but may not be the same point. The matrix M_2 is then $M_2 = B_4 \circ B_3 \circ B_2 \circ B_1$ where the matrices B_k are

$$B_i = \begin{bmatrix} 1 & -a_i/b_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -c_i/b_i & 1 & 0 \\ 0 & -d_i/b_i & 0 & 1 \end{bmatrix}, \qquad B_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -b_j/a_j & 1 & 0 & 0 \\ -c_j/a_j & 0 & 1 & 0 \\ -d_j/a_j & 0 & 0 & 1 \end{bmatrix},$$

for $i = 1, 3$ and $j = 2, 4$. The composition M_2 has the form

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bar{E}_1 & \bar{E}_2 & 0 & 0 \\ \bar{D}_1 & \bar{D}_2 & 1 & 0 \\ \bar{D}_3 & \bar{D}_4 & 0 & 1 \end{bmatrix}, \tag{12}$$

where the entries

$$\begin{aligned} \bar{D}_1 &= -\frac{c_5}{a_5} + \frac{b_5}{a_5}\frac{c_6}{b_6} - \frac{c_7}{a_7}\frac{a_6}{b_6}\frac{b_5}{a_5}, \; \bar{D}_2 = -\frac{c_1}{b_1} + \frac{c_5}{a_5}\frac{a_1}{b_1} - \frac{a_1}{b_1}\frac{b_5}{a_5}\frac{c_6}{b_6} + \frac{a_1}{b_1}\frac{b_5}{a_5}\frac{a_6}{b_6}\frac{c_7}{a_7} \\ \bar{D}_3 &= -\frac{d_5}{a_5} + \frac{b_5}{a_5}\frac{d_6}{b_6} - \frac{d_7}{a_7}\frac{a_6}{b_6}\frac{b_5}{a_5}, \; \bar{D}_4 = -\frac{d_1}{b_1} + \frac{d_5}{a_5}\frac{a_1}{b_1} - \frac{a_1}{b_1}\frac{b_5}{a_5}\frac{d_6}{b_6} + \frac{a_1}{b_1}\frac{b_5}{a_5}\frac{a_6}{b_6}\frac{d_7}{a_7} \\ \bar{E}_1 &= -\frac{b_7}{a_7}\frac{a_6}{b_6}\frac{b_5}{a_5}, \qquad \bar{E}_2 = \frac{b_7}{a_7}\frac{a_6}{b_6}\frac{b_5}{a_5}\frac{a_1}{b_1}. \end{aligned}$$

The structure of the system (1) restrict the dynamics in a very significant way.

Lemma 2.4 *In system (1) with* **(H1)** *and* **(H2)** *we must have*

$$E_2 = \bar{E}_2 = 1.$$

Proof. We start by observing, that the form of the equations (1) and (**H1**) implies that the first component is

$$\dot{x}_1 = -x_1 + \Lambda_1(x_2, x_3, x_4).$$

It follows that the first components of the target points are $a_1 = \Lambda_1(-,+,-)$, $a_2 = \Lambda_1(-,+,+)$, $a_3 = \Lambda_1(-,-,+)$, $a_4 = \Lambda_1(-,-,-)$, $a_5 = \Lambda_1(+,+,-)$, $a_6 = \Lambda_1(+,+,-)$, and $a_7 = \Lambda_1(-,+,-)$. Therefore $a_1 = a_7$ and $a_5 = a_6$. Similar considerations involving second, third and fourth component give $b_6 = b_7$, $b_1 = b_5$, $c_2 = c_3$, $c_1 = c_4$, $d_1 = d_2$ and $d_3 = d_4$. Using formulas for E_1 and $\bar{E}_1$ we get

$$E_1 = -\frac{d_1}{c_1} \qquad \text{and} \qquad \bar{E}_1 = -\frac{b_1}{a_1} \tag{13}$$

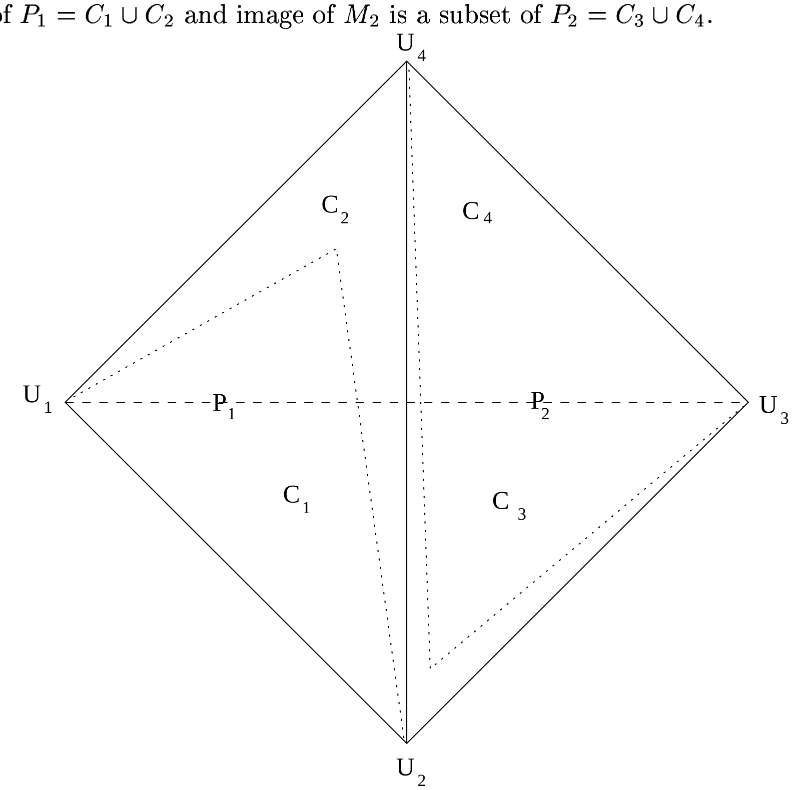
and formulas for E_2 and $\bar{E}_2$ give $E_2 = \bar{E}_2 = 1$. □

Note that Lemma 2.1 is not true with assumption **(H1')**. The assumption (**H3**) allows us to construct a simple picture of the return map on P. To see this, we determine the domains of maps $\mathbf{M}_1$ and $\mathbf{M}_2$ in P by computing time t_2 when a trajectory starting from $x \in P$ hits the plane $x_2 = 0$ and a time t_4, when that trajectory hits the plane $x_4 = 0$. If $t_4 < t_2$ then the trajectory leaves through $x_4 = 0$ and thus goes through the loop L_1. When $t_2 < t_4$ then the trajectory goes through the loop L_2. Hence the domain of the map $\mathbf{M}_1$ in P is the set of points $x \in P$ such that $t_4 < t_2$ and the domain of $\mathbf{M}_2$ in P is the set with $t_2 < t_4$. Setting $x_2(t) = 0$ and solving for t we compute

$$e^{-t_2} = \frac{-b_1}{x_2(0) - b_1} \quad \text{and} \quad e^{-t_4} = \frac{-d_1}{x_4(0) - d_1}.$$

It follows that the dividing line between the domain of $\mathbf{M}_1$ and $\mathbf{M}_2$ is a curve given by $\frac{x_2(0)}{x_4(0)} = \frac{b_1}{d_1}$. With assumption (**H3**) we get that $P \cap \{x_2 < x_4\}$ is the domain of $\mathbf{M}_1$ and $P \cap \{x_2 > x_4\}$ is the domain of $\mathbf{M}_2$ (recall that both $x_2 < 0$ and $x_4 < 0$ in P), see Figure 2.

Figure 2: Poincaré section $P = P_1 \cup P_2$. Domain of the map M_1 is $C_1 \cup C_3$, domain of M_2 is $C_2 \cup C_4$, image of M_1 is a subset of $P_1 = C_1 \cup C_2$ and image of M_2 is a subset of $P_2 = C_3 \cup C_4$.



We divide P into domains $C_1 := P_1 \cap \{x_2 < x_4\}$, $C_2 := P_1 \cap \{x_2 > x_4\}$ and $C_3 := P_2 \cap \{x_2 < x_4\}$, $C_4 := P_2 \cap \{x_2 > x_4\}$. There are 28 parameters a_j, b_j, c_j, d_j. However, the parameters $D_i, \bar{D}_i, E_i$ and $\bar{E}_i$ are related.

Lemma 2.5 *Consider system (1) with* (**H1**),(**H2**) *and* (**H3**) . *Then the constants* $D_i, \bar{D}_i$, $i = 1, \ldots, 4$ *and* $E_j, \bar{E}_j, j = 1, 2,$ *satisfy following restrictions:*

$$\begin{aligned} &E_1 = \bar{E}_1 < 0, && \bar{E}_2 = 1,\ E_2 = 1 \\ &D_1 > 0,\ \bar{D}_1 > 0 && D_2 = \frac{1 + D_1}{E_1} < 0,\ \bar{D}_2 = \frac{1 + \bar{D}_1}{\bar{E}_1} < 0 \qquad (14) \\ &D_3 < 0,\ \bar{D}_3 < 0 && D_4 = -1 + \frac{D_3}{E_1},\ \bar{D}_4 = -1 + \frac{\bar{D}_3}{\bar{E}_1} \end{aligned}$$

(15)

Proof. Direct computation using (9), (11) and (13). □

3 Periodic orbits of type $(k, 1)$

Let

$$h_{M_1} := \frac{\bar{D}_3 - \bar{E}_1}{D_3 - E_1} \frac{1}{\bar{D}_1 + \bar{D}_3/E_1}, \ h_{M_2} := \frac{D_3 - E_1}{\bar{D}_3 - \bar{E}_1} \frac{1}{D_1 + D_3/\bar{E}_1}.$$

Recall that S is the maximal invariant set in the interior of the set A under the R-flow.

Theorem 3.1 *Consider the R-flow of the system (1) with* **(H1)**,**(H2)** *and* **(H3)**.

1. *if both $D_3 - E_1 \leq 0$ and $\bar{D}_3 - \bar{E}_1 \leq 0$ then S consists of two equilibria $U_1 = (1, 0, 0, 0)$ and $U_3 = (0, 0, 1, 0)$.*
2. *if $D_3 - E_1 \leq 0$ and $\bar{D}_3 - \bar{E}_1 > 0$ then $S = \{U_1\}$.*
3. *if $D_3 - E_1 > 0$ and $\bar{D}_3 - \bar{E}_1 \leq 0$ then $S = \{U_3\}$.*
4. *Assume both $D_3 - E_1 > 0$ and $\bar{D}_3 - \bar{E}_1 > 0$. Then if*

$$h_{M_i} = k$$

and $(\bar{D}_1 + \bar{D}_3/E_1)(D_1 + D_3/\bar{E}_1) > 1$ then S consists of a unique periodic orbit with pattern consisting of k-copies of symbol i followed by a single copy of the other symbol.

Proof. We outline the proof. To prove the first part we show by direct computation that under the map $\mathbf{M}_1$ all he points in C_1 approach the point U_1. Similar argument holds for $\mathbf{M}_2$, C_2 and the point U_3. In part 2 we first show that all points leave $C_2 \cup C_4$ under some iterate of $\mathbf{M}_2$ and enter C_3 (Lemma 3.2). Then first part applies and point converges to U_1. Proof of part 3 is analogous. The most interesting is part 4. In this case, points can move between domains of $\mathbf{M}_1$ and $\mathbf{M}_2$ multiple times. We develop a pair of Lyapunov functions α and $\bar{\alpha}$ and show that the quantity h_{M_i} determines how many iterates of M_i it takes for a point to leave domain of M_i. We use index argument to show existence of periodic orbit with a given pattern..

To prove part 1, we first observe that if $D_3 \leq E_1$ then $M_1 U_3 = (D_1, D_3, 0, E_1)^T \in C_1$ and, by linearity, image of $\mathbf{M}_1$ is a subset of $C_1 \subset P_1$. Similarly, if $\bar{D}_3 \leq \bar{E}_1$, the image of $\mathbf{M}_2$ is a subset of C_4. Now we consider the stability of the boundary $\{x_4 = 0\}$ in C_1 under $\mathbf{M}_1$. Take a point $A = (A_1, A_2, 0, A_4) \in C_1$, where $A_1 > 0$ $A_2 < 0, A_4 < 0$ and $A_1 - A_2 - A_4 = 1$. Then $M_1 A = (A_1 + A_4 D_2, A_2 + A_4 D_4, 0, A_4)$. To get the point $\mathbf{M}_1(A)$ we normalize by dividing by the quantity $x_1 - x_2 - x_4$ where x_i denotes the i-th component. The fourth component of $\mathbf{M}_1(A)$ is

$$A_4' := \frac{A_4}{1 - A_4(D_4 - D_2)}.$$

Observe that by Lemma 2.5 $D_3 \leq E_1$ implies that $D_4 = (D_3 - E_1)/E_1 \geq 0$. Since $D_2 < 0$ we see that $|A_4'| < |A|$ and so the iterates are getting closer to the $x_4 = 0$ boundary in C_1. Furthermore, the second component of $\mathbf{M}_1(A)$ is smaller in absolute value then the second component of A and the first component of $\mathbf{M}_1(A)$ is larger then the first component of A. It follows that U_1 is the stable equilibrium under the iterates $\mathbf{M}_1^k$. Similar argument shows that U_3 is attracting under iterations of map $\mathbf{M}_2$.

To prove the second and third statement of the Theorem 3.1, we need a Lemma.

Lemma 3.2 *If $D_3 - E_1 > 0$ then all points, except points of the form $(u, v, 0, 0)$, leave $C_1 \cup C_3$ under some iteration of M_1; if $\bar{D}_3 - \bar{E}_1 > 0$ then all points, except points of the form $(0, 0, u, v)$, leave $C_2 \cup C_4$ under some iteration of M_2.*

Proof. We prove the statement if $\bar{D}_3 - \bar{E}_1 > 0$. The proof of other statement is analogous.

For any $A = (A_1, A_2, A_3, A_4)$ let $R(x) = \frac{A_4}{A_2}$ be the ratio of the fourth and the second component. Clearly, if $x \in C_2$ then $MM_2(x) \in C_4$. So let $A \in C_4$. Then $M_2 A = (0, A_2, A_3 + A_2 \bar{D}_2, A_4 + A_2 \bar{D}_4)$ and

$$R(M_2 A) = R(\mathbf{M}_2(A)) = \frac{A_4 + A_4 \bar{D}_4}{A_2} < R(A)$$

since $A_2 < 0, A_4 < 0$ and $\bar{D}_4 = \frac{\bar{D}_3 - \bar{E}_1}{\bar{E}_1} > 0$. Eventually $R(M_2^k(A)) < 1$ and thus $M_2^k(A) \in C_3$. □

Now we prove the second statement of Theorem 3.1. Since S is the set which attracts the interior of A all we need to show is that all points, except the boundary fixed points $(0,0,u,v)$ will leave the set C_4. Then they enter C_3 in the domain of the map $\mathbf{M}_1$ and and converge to U_1 by the first statement of the Theorem 3.1. The proof of the third statement of Theorem 3.1 is analogous to the second part.

We postpone the proof of the fourth part to the end of this section.

Lemma 3.3 *The set of points $A = (0, A_2, A_3, A_4) \in C_3$ which are mapped by $\mathbf{M}_1^n$ onto the points with $x_2 = x_4$ satisfy the equation*

$$p_n: \quad A_2 + n(D_3 - E_1)A_3 + (-1 + nD_4)A_4 = 0. \tag{16}$$

The set of points $A = (A_1, A_2, 0, A_4) \in C_2$ which are mapped by $\mathbf{M}_2^n$ onto the points with $x_2 = x_4$ satisfy the equation

$$\bar{p}_n: \quad A_4 + n(\bar{D}_3 - \bar{E}_1)A_1 + (-1 + nD_4)A_2 = 0. \tag{17}$$

Proof. Direct computation using Lemma 2.5. □

We shall associate to every point $A = (0, A_2, A_3, A_4) \in C_3$ a number α

$$\alpha(A) = \frac{A_4 - A_2}{(D_3 - E_1)A_3 + D_4 A_4}. \tag{18}$$

It is easy to check that with such α the point A lies on on the plane given by the equation

$$A_2 + \alpha(D_3 - E_1)A_3 + (-1 + \alpha D_4)A_4 = 0.$$

We may define in the same way the number $\bar{\alpha}$ for any point $A \in C_2$

$$\bar{\alpha}(A) = \frac{A_2 - A_4}{(\bar{D}_3 - \bar{E}_1)A_1 + \bar{D}_4 A_2}. \tag{19}$$

Such a point lies on the plane $A_4 + \bar{\alpha}(\bar{D}_3 - \bar{E}_1)A_1 + (-1 + \bar{\alpha}D_4)A_2 = 0$. We shall use numbers α and $\bar{\alpha}$ to describe a position of a point $x \in P$ under the iterations of the maps $\mathbf{M}_1$ and $\mathbf{M}_2$. Observe, that since both α and $\bar{\alpha}$ are defined as a ratio, we have

$$\alpha(\mathbf{M}_2(x)) = \alpha(M_2 x), \qquad \bar{\alpha}(\mathbf{M}_1(x)) = \bar{\alpha}(M_1 x).$$

Lemma 3.4 *Assume $D_3 - E_1 > 0$ and $\bar{D}_3 - \bar{E}_1 > 0$. Then for any $x \in C_2 \cup C_4$ such that $\mathbf{M}_2(x) \in C_3$ we have $\alpha(\mathbf{M}_2(U_1)) \geq \alpha(\mathbf{M}_2(x))$. For any $x \in C_1 \cup C_3$ such that $\mathbf{M}_1(x) \in C_2$ we have $\bar{\alpha}(\mathbf{M}_1(U_3)) \geq \bar{\alpha}(\mathbf{M}_1(x))$.*

Proof. We prove the result for α, the proof for $\bar{\alpha}$ being analogous. A direct computation shows that

$$\alpha(\mathbf{M}_2(U_1)) = \frac{\bar{D}_3 - \bar{E}_1}{(D_3 - E_1)(\bar{D}_1 + \bar{D}_3/E_1)}.$$

Let $A = (A_1, A_2, 0, A_4) \in C_2$ be an arbitrary point. Then

$$\alpha(\mathbf{M}_2(A)) = \frac{A_1(\bar{D}_3 - \bar{E}_1) + A_2(\bar{D}_4 - 1) + A_4}{(D_3 - E_1)(A_1\bar{D}_1 + A_2\bar{D}_2) + D_4(A_1\bar{D}_3 + A_2\bar{D}_4 + A_4)}.$$

Using Lemma 2.5 repeatedly we get

$$\alpha(\mathbf{M}_2(A)) = \frac{(\bar{D}_3 - \bar{E}_1)(A_1 + A_2/\bar{E}_1) + A_4 - A_2}{(D_3 - E_1)(\bar{D}_1 + \bar{D}_3/E_1)(A_1 + A_2/\bar{E}_1) + D_4 A_4}.$$

Since $A_2 < 0, D_4 < 0$ and $A_4 - A_2 < 0$ by the fact $x \in C_2$, the result follows.

For $A = (0, A_2, A_3, A_4) \in C_4$ we get by a similar computation

$$\alpha(\mathbf{M}_2(A)) = \frac{(\bar{D}_3 - \bar{E}_1)A_2/\bar{E}_1 + A_4 - A_2}{(D_3 - E_1)(\bar{D}_1 + \bar{D}_3/E_1)A_2/\bar{E}_1 + D_4 A_4 + (D_3 - E_1)A_3}.$$

Observe that $D_4 < 0$ and $D_3 - E_1 > 0$ by assumption and $A_4 - A_2 < 0, A_4 < 0, A_3 > 0$ by the fact that $A \in C_4$. The result now follows. □

Lemma 3.5 *If $h_{M_i} \leq n$ then for arbitrary $x \in S$ there may be at most n symbols 'i' in a row in a pattern $z(x)$.*

Proof. We shall prove the Lemma for h_{M_1}. From Lemma 3.4 we have that $\alpha(\mathbf{M}_2(x)) \leq \alpha(\mathbf{M}_2(U_1)) = h_{\mathbf{M}_1}$ for any x in the domain of $\mathbf{M}_2$ such that $\mathbf{M}_2(x) \in C_3$. From the definition of α it follows that if $h_{M_1} \leq n$ it will take at most n iterates of $\mathbf{M}_1$ to take a point $x \in \mathbf{M}_2(C_2 \cup C_4)$ back to $C_2 \cup C_4$. Finally, by Lemma 3.2 if $x \in S$ then it must be in the image of $\mathbf{M}_2$. □

Let

$$K_\alpha = (0, -\alpha(D_3 - E_1), 1, 0) \qquad Q_\alpha = (0, \alpha D_4 - 1, 0, -1)$$

be the intersection of the plane $p_\alpha : A_2 + \alpha(D_3 - E_1)A_3 + (-1 + \alpha D_4)A_4 = 0$ with $\{x_4 = x_1 = 0\}$ and $\{x_3 = x_1 = 0\}$, respectively. The analogous points using $\bar{\alpha}$ will be denoted by $\bar{K}_\alpha$ and $\bar{Q}_\alpha$.

Proof of Theorem 1.3

Assume $x \in C_1 \cup C_3$ such that $\mathbf{M}_1^{l+1}(x) \in C_2$ and $\mathbf{M}_1^l(x) \in C_1$. By Lemma 3.5 for every $x \in S \cap (C_1 \cup C_3)$ there is such an l. Then, by the definition of planes p_n and $\bar{p}_n$ $\mathbf{M}_1^l(x)$ is in domain bounded by $x_2 = x_4$ line and line from U_1 to the point Q_1. We call this domain T. Similarly, if $x \in C_2 \cup C_4$ such that $\mathbf{M}_2^{l+1}(x) \in C_3$ and $\mathbf{M}_2^l(x) \in C_4$, then $\mathbf{M}_2^l(x)$ is in the domain bounded by $x_2 = x_4$ and a line from U_3 to $\bar{Q}_1$. We call this domain $\bar{T}$, see Figure 3. Let $X := [0, -1/2, 0, -1/2]^T$. Similar idea as in the proof of Lemma 3.5 shows that

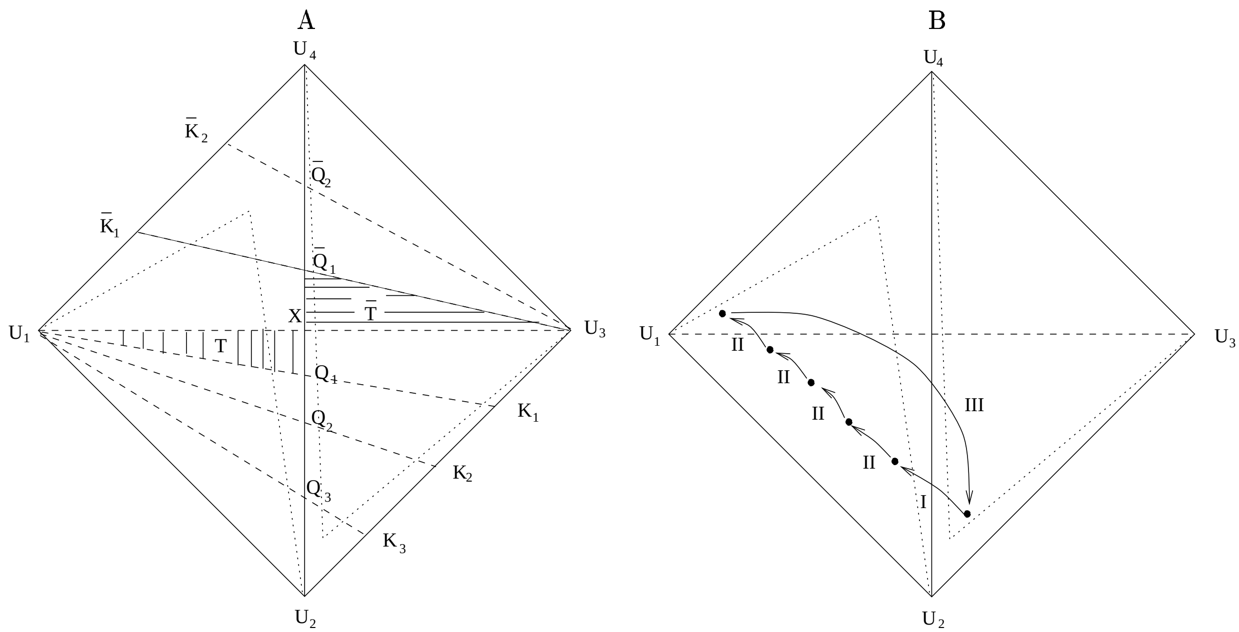


Figure 3: A. Sets T and $\bar{T}$ with points X, Q_i, K_i and $\bar{Q}_i$. B. A periodic orbit of type $(5, 1)$ on the Poincaré section P. Dotted triangle on the left is the image of $\mathbf{M}_1$ and on the right image of $\mathbf{M}_2$. Type of transition is denoted by the Roman numerals (see proof of Theorem 1.2).

for any $x \in \bar{T}$ we have

$$\alpha(M_2 x) \leq \alpha(M_2 X) \tag{20}$$

and for any $x \in T$ we have $\bar{\alpha}(M_1 x) \leq \bar{\alpha}(M_1 X)$. We compute $\alpha(M_2 X)$. Clearly, $M_2 X = -1/2(0, 1, \bar{D}_2, \bar{D}_4 + 1)$ and

$$\alpha(M_2 X) = \frac{(\bar{D}_3 - \bar{E}_1)1/\bar{E}_1}{(D_3 - E_1)(\bar{D}_2 + D_4(\bar{D}_4 + 1))}.$$

By using Lemma 2.5 repeatedly we get $\alpha(M_2X) = \frac{(\bar{D}_3-\bar{E}_1)}{(D_3-E_1)(1+\bar{D}_1+\bar{D}_3/E_1)}$. By symmetry we get that $\bar{\alpha}(M_1X) = \frac{(D_3-E_1)}{(\bar{D}_3-\bar{E}_1)(1+D_1+D_3/\bar{E}_1)}$. Multiplying the two expressions together, we get

$$\bar{\alpha}(M_1X)\alpha(M_2X) = \frac{1}{(1+\bar{D}_1+\bar{D}_3/E_1)(1+D_1+D_3/\bar{E}_1)} < 1. \tag{21}$$

It is not possible that both $\bar{\alpha}(M_1X)$ and $\alpha(M_2X)$ are bigger then one. Hence if there is an $x \in S$ which have multiple symbols 1 in $z(x)$ we must have that $\alpha(M_2X) > 1$. Then there cannot be multiple symbols 2 in $z(x)$ in view of (20) and (21). The analogous argument applies to $x \in S$ which have multiple symbols 2 in $z(x)$ □

From now on we will assume without loss of generality that $h_{M_2} < 1$.

Lemma 3.6 *Consider the R-flow of the system (1) in* $\mathbf{R}^4$ *with attracting set A. Assume* **(H1)**,**(H2)**, **(H3)** *and that*

$$h_{M_1} = k \qquad \text{and} \qquad (\bar{D}_1 + \bar{D}_3/E_1)(D_1 + D_3/\bar{E}_1) > 1.$$

Then for all $x \in C_3$ *with* $l-1 \le \alpha(x) \le l$ *and* $2 \le l \le k-1$,

1. $\alpha(\mathbf{M}_2\mathbf{M}_1^l(x)) > \alpha(x)$
2. *If* $\alpha(x) = k-1$ *then* $\alpha(\mathbf{M}_2\mathbf{M}_1^k(x)) > k-1$.

Proof. Given $x \in C_3$, there is unique point point K_β with $\beta = \alpha(x)$ and the unique point Q_β with $\beta = \alpha(x)$. We will show that

$$\alpha(\mathbf{M}_2\mathbf{M}_1^l(K_\beta)) > \beta \qquad \text{and} \qquad \alpha(\mathbf{M}_2\mathbf{M}_1^l(Q_\beta)) > \beta. \tag{22}$$

Since the map $\mathbf{M}_2\mathbf{M}_1^l$is linear, this will prove the first part of the Lemma.

First we compute $\mathbf{M}_2\mathbf{M}_1^l(K_\beta)$. Since $M_1K_\beta = (D_1, -\beta(D_3-E_1)+D_3, 0, E_1)^T$ and

$$M_1^{l-1}((A_1, A_2, 0, A_4)^T) = (A_1 + (l-1)D_2A_4, A_2 + (l-1)D_4A_4, 0, A_4)^T \tag{23}$$

we get, using Lemma 2.5, that $M_1^lK_\beta = (lD_1 + l - 1, (D_3-E_1)(l-\beta)+E_1, 0, E_1)^T$. Then $M_2M_1^lK_\beta =$

$$\begin{aligned}&(0, (lD_1+l-1)\bar{E}_1 + (D_3-E_1)(l-\beta)+E_1, (lD_1+l-1)\bar{D}_1 + \\ &\bar{D}_2((D_3-E_1)(l-\beta)+E_1), (lD_1+l-1)\bar{D}_3 + \bar{D}_4((D_3-E_1)(l-\beta)+E_1)+E_1)^T.\end{aligned} \tag{24}$$

We now compute $\alpha(\mathbf{M}_2\mathbf{M}_1^l(K_\beta))$:

$$\alpha(\mathbf{M}_2\mathbf{M}_1^k(K_\beta)) = \frac{(\bar{D}_3-\bar{E}_1)(lD_1+l-1) + \bar{D}_4((D_3-E_1)(l-\beta)+E_1) - (D_3-E_1)(l-\beta)}{Z}$$

where Z is given by $Z = (D_3-E_1)((lD_1+l-1)\bar{D}_1 + \bar{D}_2((D_3-E_1)(l-\beta)+E_1) + D_4((lD_1+l-1)\bar{D}_3 + \bar{D}_4((D_3-E_1)(l-\beta)+E_1)+E_1)$. Since $\bar{D}_4 = \frac{\bar{D}_3-\bar{E}_1}{\bar{E}_1}$ and $D_4 = \frac{D_3-E_1}{E_1}$ we can simplify the ratio to

$$\frac{(\bar{D}_3-\bar{E}_1)(lD_1+l-1+((D_3-E_1)(l-\beta)+E_1)/\bar{E}_1) - (D_3-E_1)(l-\beta)}{(D_3-E_1)((lD_1+l-1)(\bar{D}_1+\bar{D}_3/E_1) + (\bar{D}_2+\bar{D}_4/E_1)((D_3-E_1)(l-\beta)+E_1)) + D_3 - E_1}.$$

We compute that $\bar{D}_2 + \bar{D}_4/E_1 = (\bar{D}_1 + \bar{D}_3/E_1 + 1 - \bar{E}_1/E_1)/\bar{E}_1$ and so the denominator is

$$\begin{aligned}&(D_3-E_1)(lD_1+l-1+((D_3-E_1)(l-\beta)+E_1)/\bar{E}_1)(\bar{D}_1+\bar{D}_3/E_1) \\ &\quad +(D_3-E_1)(1+(1/\bar{E}_1-1/E_1)((D_3-E_1)(l-\beta)+E_1)).\end{aligned}$$

Finally, since $E_1 = \bar{E}_1$, we have

$$\alpha(\mathbf{M}_2\mathbf{M}_1^l(K_\beta)) = \frac{(\bar{D}_3-\bar{E}_1)U - (D_3-E_1)(l-\beta)}{(D_3-E_1)(\bar{D}_1+\bar{D}_3/E_1)U + (D_3-E_1)}, \tag{25}$$

where $U := l(D_1+1)+D_4(l-\beta)$, and where we used that $D_4 = (D_3 - E_1)/E_1$. Observe that since $D_4 > -1$ and $l-\beta \leq 1$ we have $U > l(D_1+1)-1 > 0$ since $l \geq 2$. Since all other terms in the denominator are clearly positive by Lemma 2.5 and the assumption $D_3 - E_1 > 0$, the inequality $\alpha(\mathbf{M}_2\mathbf{M}_1^l(K_\beta)) > \beta$ becomes

$$(\bar{D}_3 - \bar{E}_1)U - (D_3 - E_1)(l-\beta) > \beta(D_3 - E_1)(\bar{D}_1 + \bar{D}_3/E_1)U + \beta(D_3 - E_1).$$

After collecting terms and dividing by $D_3 - E_1$ we get

$$\frac{\bar{D}_3 - \bar{E}_1}{D_3 - E_1} - \beta(\bar{D}_1 + \bar{D}_3/E_1) > \frac{l}{U}. \tag{26}$$

Now we do a similar computation with Q_β. Since $M_1 Q_\beta = (-D_2, -\beta D_4 - 1 - D_4, 0, -1)^T$ and using (23) and Lemma 2.5, we get that $M_1^l Q_\beta = (-lD_2, D_4(\beta - l) - 1, 0, -1)^T$. Then $M_2 M_1^l Q_\beta =$

$$(0, -lD_2\bar{E}_1 + D_4(\beta - l) - 1, -lD_2\bar{D}_1 + \bar{D}_2(D_4(\beta - l) - 1), -lD_2\bar{D}_3 + \bar{D}_4(D_4(\beta - l) - 1) - 1)^T,$$

and $\alpha(\mathbf{M}_2\mathbf{M}_1^l(Q_\beta)) =$

$$\frac{(\bar{D}_3 - \bar{E}_1)(-lD_2) + \bar{D}_4((\beta - l)D_4 - 1) + (l-\beta)D_4}{(D_3 - E_1)(-lD_2\bar{D}_1 + \bar{D}_2(D_4(\beta - l) - 1) + D_4(-lD_2\bar{D}_3 + \bar{D}_4(D_4(\beta - l) - 1)}$$

This can be expressed as

$$\frac{(\bar{D}_3 - \bar{E}_1)(-lD_2 + ((\beta - l)D_4 - 1)/\bar{E}_1) + (l-\beta)D_4}{(D_3 - E_1)(-lD_2 + ((\beta - l)D_4 - 1)/\bar{E}_1)(\bar{D}_1 + \bar{D}_3/E_1) + (1/\bar{E}_1 - 1/E_1)((\beta - l)D_4 - 1) - D_4}.$$

We factor out $-1/E_1$ from both the numerator and the denominator and use Lemma 2.5 and the fact that $E_1 = \bar{E}_1$ to get

$$\alpha(\mathbf{M}_2\mathbf{M}_1^l(Q_\beta)) = \frac{(\bar{D}_3 - \bar{E}_1)V - (l-\beta)(D_3 - E_1)}{(D_3 - E_1)(\bar{D}_1 + \bar{D}_3/E_1)V + (D_3 - E_1)}, \tag{27}$$

where $V := l(D_1+1) + (l-\beta)D_4 + 1$. Observe that $V = U + 1 > 0$. The equation $\alpha(\mathbf{M}_2\mathbf{M}_1^l(Q_\beta)) > \beta$ becomes, after collecting terms,

$$\frac{\bar{D}_3 - \bar{E}_1}{D_3 - E_1} - \beta(\bar{D}_1 + \bar{D}_3/E_1) > \frac{l}{V}. \tag{28}$$

Since $V > U$ to prove (22) and thus the first part of the Theorem, it is enough to prove (26). Using Lemma 2.5 we have

$$U = l(D_1 + D_3/E_1) - \beta D_4 > l(D_1 + D_3/E_1)$$

since $D_4 > 0$ by assumption $D_3 - E_1 > 0$. Therefore $\frac{l}{U} < 1/(D_1 + D_3/E_1)$ By assumption we have $h_{M_1} = k$. Using formula for h_{M_1} we get that

$$(\bar{D}_1 + \bar{D}_3/E_1)(k-\beta) > \frac{1}{D_1 + D_3/E_1} \tag{29}$$

implies (26). Since $\beta \leq k-1$ we see that $(D_1 + D_3/E_1)(\bar{D}_1 + \bar{D}_3/\bar{E}_1) > 1$ implies (29) and hence (26).

In order to prove the second statement of the Lemma we compute $\alpha(\mathbf{M}_2\mathbf{M}_1^k(Q_{k-1}))$ and $\alpha(\mathbf{M}_2\mathbf{M}_1^k(K_{k-1}))$. The computation is analogous and the number l in (26) becomes k and the number β becomes $k-1$. Then inequality (29) is still valid with $\beta = k-1$. The result again follows from the assumption $(D_1 + D_3/E_1)(\bar{D}_1 + \bar{D}_3/\bar{E}_1) > 1$. □

Completion of proof of Theorem 3.1

Recall that we assumed that $h_{M_2} < 1$. Then by Theorem 1.3 for any x in the invariant set the pattern $z(x)$ may not contain more than one symbol "2" in a row. Observe that by the Lemma 3.6 the number α serves as a Lyapunov function. It follows that only admissible sequence $z(x)$ for $x \in S$, is a periodic sequence of k copies of "1" followed by a single "2".

Let G be the region $G := \{x \in C_3 \mid k-1 < \alpha(x) < k\}$. By Lemma 3.6.3, if $\alpha(x) = k-1$ then $\alpha(\mathbf{M}_2\mathbf{M}_1^k(x)) > k-1$. Also, by Lemma 3.4 if $\alpha(x) = k$, then

$$\alpha(\mathbf{M}_2\mathbf{M}_1^k(x)) \leq h_{M_1} = k.$$

It follows that $\mathbf{M}_2\mathbf{M}_1^k(G) \subset G$ and by Brouwer fixed point Theorem $\mathbf{M}_2\mathbf{M}_1^k$ has a fixed point in G.

We will use fixed point index and the linearity of $\mathbf{M}_2\mathbf{M}_1^k$ to show that there is unique attracting fixed point of $\mathbf{M}_2\mathbf{M}_1^k$ in G. Since $\mathbf{M}_2\mathbf{M}_1^k(\partial G) \subset G$ the fixed point index of the map $\mathbf{M}_2\mathbf{M}_1^k$ along the boundary of G is 1. It is easy to check that the vector $(0,0,1,-E_1)$ is an eigenvector of both M_1 and M_2 with eigenvalue 1. It follows that it is also eigenvector of the composition $M_2M_1^k$. The vector $(1,-E_1,1,-E_1)$ is in the null space of $M_2M_1^k$. Since the region G maps strictly to itself, the map $\mathbf{M}_2\mathbf{M}_1^k$ does not have a continuum of fixed points, since by linearity this would correspond to line of equilibria. Also, eigenplane corresponding to a pair of complex eigenvalues of $M_2M_1^k$ cannot intersect G for the same reason.

It follows that the fixed points in G are isolated. Since neither vector $(0,0,1,-E_1)$ nor vector $(1,-E_1,1,-E_1)$ are in P, there are at most two fixed points of $M_2M_1^k$ in P and hence in G. By Lemma 2.3 the index of each of these fixed points is ± 1. Since the index along the boundary of G is 1, the only possibility is that there is unique fixed point in G with index 1. It must be an attracting fixed point since the boundary of G is mapped inside G. □

Proof of Theorem 1.2

Theorem 3.1 shows that under which conditions of $D_i, \bar{D}_i$ and $E_1 = \bar{E}_1$ there is a $(k,1)$ periodic orbit in the R-flow. We need to show that there is an appropriate choice of functions Λ_i, or, which is equivalent, the choice of target points $T_1 - T_7$, subject to restrictions in Lemma 2.4, such that it corresponds to a $(k,1)$ periodic orbit in the flow of (1).

We use Theorem 2.2 to check that all transitions are expanding. The only orthant where we need to check condition (8) is the splitting orthant $\mathcal{O}$ with signature $(+,-,+,-)$. Observe that for the periodic orbit of the type $(k,1)$ there is one transition from C_3 to C_1, $k-1$ transitions from C_1 to $C_1 \cup C_2$ and one transition from $C2$ to $C3$ (see Figure 3, part B). We denote these transitions by Roman numerals I, II, III. Observe that for transition I, which corresponds in the splitting orthant $\mathcal{O}$ to the transition from wall $x_1 = 0$ to the wall $x_4 = 0$, condition (8) is $b_1x_2 + c_1x_3 > 0$. For transition II ($\{x_3 = 0\} \to \{x_4 = 0\}$) condition (8) is $a_1x_1 + b_1x_2 > 0$, and, finally, for transition III ($\{x_3 = 0\} \to \{x_2 = 0\}$) it is $a_1x_1 + d_1x_4 > 0$. Using **(H3)** we get three conditions **(I.)** $\frac{b_1}{a_1}x_2 + x_3 > 0$ (in C_3), **(II.)** $\frac{b_1}{a_1}x_2 + x_1 > 0$ (in C_1), and **(III.)** $\frac{b_1}{a_1}x_4 + x_1 > 0$ (in C_2). Observe that if $\frac{b_1}{a_1}$ is sufficiently small then the periodic orbit of type $(k,1)$ under R-flow is in the region in P described by the equations above. Because **(H3)**, the second condition in (8) is also satisfied for any such choice of a_1 and b_1.

A periodic orbit of the type $(k,1)$ exists in R-flow for a choice of constants $D_i, \bar{D}_i, E_1$, $i = 1, \ldots, 4$, satisfying Theorem 3.1.4. Fixing these constants we make the ratio $\frac{b_1}{a_1}$ sufficiently small and select target points so that second condition in (8) is satisfied for all of them. Then there is a periodic orbit of the type $(k,1)$ in the flow of (1). □

4 Symmetric invariant set

In this section we assume the symmetry

$$(a_{i+3}, b_{i+3}, c_{i+3}, d_{i+3}) = (c_i, d_i, a_i, b_i), \qquad i = 2,3,4. \tag{30}$$

This assumption implies that, in addition to $E_2 = \bar{E}_2 = 1$ and $E_1 = \bar{E}_1$ we have

$$D_i = \bar{D}_i \qquad i = 1, \ldots, 4. \tag{31}$$

The invariant set S is invariant under the symmetry $(x,y,z,w) \to (z,w,x,y)$.

The main goal of this section is to prove following Theorem, which implies Theorem 1.4.

Theorem 4.1 *Consider the R-flow associated to (1) in* $\mathbf{R}^4$ *with the attracting set* A. *Assume* **(H1)**,**(H2)**,**(H3)** *and (30). Then*

1. *if* $D_3 \le E_1$ *then* S *consists of two equilibria* $U_1 = (1,0,0,0)$ *and* $U_3 = (0,0,1,0)$.

2. *if* $D_3 > E_1$ *then the set* S *is unique periodic orbit of type* $(1,1)$.

We postpone the proof to the end of the section. We denote

$$\mathbf{F}_l := \mathbf{M}_2 \circ \mathbf{M}_1^l.$$

Let $Z_l := \{x \in C_3 \mid l-1 \le \alpha(x) \le l\}$ be the domain of the map F_l for all $l \le k$. Consider a composition of maps of the form $\mathbf{F}_{l_s}^{k_s} \dots \mathbf{F}_{l_1}^{k_1}$. We say that a fixed point of this map is *admissible* if it lies in the domain Y

$$Y := Z_{l_1} \cap F_{l_1}^{-k_1}(Z_{l_2}) \cap F_{l_1}^{-k_1}(F_{l_2}^{-k_2}(Z_{l_3})) \cap \dots \cap F_{l_1}^{-k_1}(\dots(F_{l_{s-1}}^{-k_{s-1}}(Z_{l_s}))).$$

Observe that only admissible fixed points represent a periodic orbit in the R-flow.

To prove the second part of the Theorem we shall investigate in detail the maps $\mathbf{F}_l$ on the (admissible) domains Z_l. We first show that the maps $\mathbf{F}_l$ for all $l \ge 2$ have no equilibria in Z_l. This will be done by the direct computation of the eigenvectors of the corresponding matrices F_l. These correspond to the fixed points of the maps $\mathbf{F}_l$ associated to the R-flow.

In the second step we use a fixed point index argument to show that compositions of maps $\mathbf{F}_l$ do not admit admissible fixed points.

Lemma 4.2 *Assume* **(H1)**,**(H2)** *and* **(H3)** *and (30). The characteristic polynomial of* $M_2 M_1^l$ *is*

$$x(x-1)(x^2 - x(l(D_1 + D_3/E_1)^2 + 2) + 1).$$

The eigenvalue-eigenvector pairs of $M_2M_1^l$ *are* $(0; (1, -E_1, 1, -E_1))$, $(1; (0,0,1,-E_1))$, $(\gamma_l + \beta_l; (0, a_2^l, a_3^l, a_4^l))$, *and* $(\gamma_l - \beta_l; (0, r_2^l, r_3^l, r_4^l))$.

Furthermore,

1. $\gamma_l + \beta_l > 1$, $sgn(a_2^l) = sgn(a_4^l)$ *and* $sgn(a_3^l) = \begin{cases} sgn(a_2^l) & \text{if } l \ge 2 \\ -sgn(a_2^l) & \text{if } l = 1 \end{cases}$

2. $\gamma_l - \beta_l < 1$ *and* $sgn(r_3^l) = sgn(r_4^l) = sgn(r_2^l)$.

Proof. We first compute the matrix $M_2 M_1^l$. A straightforward calculation gives

$$M_1^l = \begin{bmatrix} 1 & 0 & D_1 + (l-1)D_2E_1 & lD_2 \\ 0 & 1 & D_3 + (l-1)D_4E_1 & lD_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_1 & 1 \end{bmatrix}, \quad M_2M_1^l = \begin{bmatrix} 0 & 0 & 0 & 0 \\ E_1 & 1 & S_2 & l(D_2E_1 + D_4) \\ D_1 & D_2 & S_3 & l(D_1D_2 + D_2D_4) \\ D_3 & D_4 & S_4 & l(D_2D_3 + D_4^2) + 1 \end{bmatrix},$$

where $S_2 = E_1(D_1 + (l-1)D_2E_1) + D_3 + (l-1)D_4E_1$, $S_3 = D_1(D_1 + (l-1)D_2E_1) + D_2(D_3 + (l-1)D_4E_1)$ and $S_4 = D_3(D_1 + (l-1)D_2E_1) + D_4(D_3 + (l-1)D_4E_1) + E_1$.

Using Lemma 2.5 one gets that $M_2M_1^l$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ E_1 & 1 & l(E_1D_1 + D_3) & l(D_1 + D_3/E_1) \\ D_1 & (1+D_1)/E_1 & l(D_1(D_1 + D_3/E_1) + D_3/E_1) - l + 1 & lD_2(D_1 + D_3/E_1 - 1) \\ D_3 & -1 + D_3/E_1 & l(D_3(D_1 + D_3/E_1) - (D_3 - E_1)) & W \end{bmatrix},$$

where $W = l(D_3/E_1(D_1 + D_3/E_1) - D_3/E_1) + l + 1$ It is easy to check that the vector $(1, -E_1, 1, -E_1)$ is in the null space of the matrix $M_2M_1^l$. The other three eigenvectors can be found from the reduced matrix, which we get by dropping the first column and the first row in $M_2M_1^l$. We used MAPLE symbolic manipulation package to compute the characteristic polynomial of the reduced matrix as

$$(x-1)(x^2 - x(l(D_1 + D_3/E_1)^2 + 2) + 1).$$

Computing eigenvalues we get eigenvalue 1 and

$$\gamma_l \pm \beta_l := 1 + \frac{l}{2}(D_1 + D_3/E_1)^2 \pm \frac{l}{2}(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l}.$$

It is easy to see that $\gamma_l + \beta_l > 1$ and $\gamma_l - \beta_l < 1$.

For the Poincaré map in the R-flow the eigenvector with the eigenvalue $\gamma_l + \beta_l$ corresponds to an attractor, with 1 to a saddle and with $\gamma_l - \beta_l$ to a repeller.

Now we compute the eigenvector $a^l := (0, a_2^l, a_3^l, a_4^l)$ corresponding to the eigenvalue $\gamma_l + \beta_l$. We fix $a_4^l = 1$ and compute, using MAPLE, the component $a_3^l := n_3/d$. Even though n_3 and d depend on l we shall not use superscripts in order not to overburden the reader with notation. The denominator d can be written as

$$d = E_1(l(D_1E_1 + D_3)(D_1D_3 + E_1) + (l-1)(D_3 - E_1)^2).$$

Since $l \geq 1$ using Lemma 2.5 we get

$$d < 0 \qquad \text{for all } l. \tag{32}$$

The numerator n_3 can be written as

$$\frac{1}{2}(1 + D_1)E_1^2(2 - 2D_3/E_1 + l(D_1^2 - (D_3/E_1)^2 + 2(D_3/E_1 - 1)) + l(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l}).$$

The third component a_3 of a will be negative if $n_3 > 0$. This happens if, and only if, $E > 0$, where

$$E := 2 - 2D_3/E_1 + l(D_1^2 - (D_3/E_1)^2 + 2(D_3/E_1 - 1)) + l(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l}).$$

We show now that $E < 0$ for all $l \geq 2$ and $E > 0$ for $l = 1$. Since $D_4 = D_3/E_1 - 1$, using Lemma 2.5 we get

$$E = l(D_1 + D_3/E_1)(\sqrt{(D_1 + D_3/E_1)^2 + 4/l} + D_1 - D_3/E_1) + 2(l-1)D_4. \tag{33}$$

We first observe that $\sqrt{(D_1 + D_3/E_1)^2 + 4/l} + D_1 - D_3/E_1 > 0$. Indeed, squaring the equation

$$\sqrt{(D_1 + D_3/E_1)^2 + 4/l} > D_3/E_1 - D_1 \tag{34}$$

this is equivalent to $4D_1D_3/E_1 + 4/l > 0$, which is satisfied by Lemma 2.5. It follows immediately from (33) that if $l = 1$ then $a_3^l < 0$ and hence

$$sgn(a_3^1) \neq sgn(a_4^1).$$

Let k be an integer such that $\frac{1}{k+1} < D_1 + D_3/E_1 \leq \frac{1}{k}$. Observe that the number k is related to the numbers $h_{M_1} = h_{M_2}$, see Lemma 3.5 and (31). We show that for $l \geq 2$ and $k \geq l$ $E < 0$. We start by proving this fact for $l \geq 2$, $k = l$. In this case we must have $D_1 + D_3/E_1 = \frac{1}{k} = \frac{1}{l}$. Using this and $D_4 = -1 + D_3/E_1$ the inequality $E < 0$ becomes $\sqrt{1/l^2 + 4/l} + D_1 - D_3/E_1 + 2(l-1)D_3/E_1 < 2(l-1)$, which is equivalent to

$$\sqrt{1/l^2 + 4/l} + D_1 + D_3/E_1 + 2(l-2)D_3/E_1 < 2(l-1).$$

We replace $D_1 + D_3/E_1 = 1/l$, and using the estimate

$$D_3/E_1 < 1/l, \tag{35}$$

we get that the inequality is implied by

$$\sqrt{1/l^2 + 4/l} + \frac{1}{l} + \frac{2(l-2)}{l} \leq 2(l-1). \tag{36}$$

Observe, that we do not need sharp inequality since the inequality in (35) is sharp. Squaring, and multiplying out we get

$$\frac{1 + 4l}{l^2} \leq \frac{4l^4 - 8l^3 + 28l^2 - 24l + 9}{l^2}$$

which will be satisfied, if and only if, $l^4 - 4l^3 + 7l^2 - 7l + 2 \geq 0$. The polynomial has two real roots, $r_1 = 2$ and $r_2 \approx 0.430159709$. Thus for $l \geq 2$ and $D_1 + D_3/E_1 = k = l$ we have $E < 0$.

Now we show that $E < 0$ for $l \geq 2$ and $D_1 + D_3/E_1 < 1/k \leq 1/l$ for any $k \geq l$. We estimate

$$\begin{aligned}
E &= l(D_1 + D_3/E_1)(\sqrt{(D_1 + D_3/E_1)^2 + 4/l} + D_1 - D_3/E_1) + 2(l-1)D_4 \\
&< l/k(\sqrt{1/k^2 + 4/l} + D_1 - D_3/E_1) - 2(l-1) + 2(l-1)D_3/E_1 \\
&\leq \sqrt{1/l^2 + 4/l} + D_1 + D_3/E_1 + 2(l-2)D_3/E_1 - 2(l-1) \\
&< \sqrt{1/l^2 + 4/l} + 1/l + 2(l-2)/l - 2(l-1) < 0.
\end{aligned}$$

The last inequality is the consequence of (36). Thus $E < 0$ for all $l \geq 2$ and all values of $k \geq l$. Therefore $sgn(a_3^l) = sgn(a_4^l)$ for all $l \geq 2$.

Now we show that $sgn(r_3^l) = sgn(r_4^l)$ for all l. We set $r_4^l = 1$. Using MAPLE we get $r_3^l = q_3/d$ with $q_3 =$

$$\frac{1}{2}(1 + D_1)E_1^2(2 - 2D_3/E_1 + l(D_1^2 - (D_3/E_1)^2 + 2(D_3/E_1 - 1)) - l(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l}).$$

Since $d < 0$ we want to show that $q_3 < 0$. Similarly as in (33) this is equivalent to

$$l(D_1 + D_3/E_1)(D_1 - D_3/E_1 - \sqrt{(D_1 + D_3/E_1)^2 + 4/l}) + 2(l-1)D_4 < 0.$$

Using (34) we see that $D_1 - D_3/E_1 - \sqrt{(D_1 + D_3/E_1)^2 + 4/l} < 0$; since $D_4 < 0$ the inequality $q_3 < 0$ is always satisfied. Hence $sgn(r_3^l) = sgn(r_4^l)$ for all l.

We show that $sgn(r_2^l) = sgn(r_4^l)$. Again using MAPLE, we set $r_4^l = 1$ and $r_2^l = q_2/d$, we have

$$\begin{aligned}
q_2 &= lE_1^3(D_1^2 D_3/E_1 + D_1 - D_1 D_3/E_1 + 2D_1 D_3^2/E_1^2 + D_3/E_1 - D_3^2/E_1^2 + D_3^3/E_1^3) - E_1^3 \\
&\quad + D_3 E_1^2 - 1/2(D_3 - E_1)E_1^2(l(D_1 + D_3/E_1)^2 + 2 - l(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l})
\end{aligned} \tag{37}$$

This can be rewritten as

$$\begin{aligned}
&E_1^3(l((D_1 + D_3/E_1) + D_3/E_1(D_1^2 + 2D_1 D_3/E_1 + D_3^2/E_1^2) - D_3/E_1(D_1 + D_3/E_1)) \\
&\quad + D_4 - \frac{D_4}{2}(l(D_1 + D_3/E_1)^2 + 2 - l(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l}))
\end{aligned}$$

where we used that $D_4 = (D_3 - E_1)/E_1$. Further, we can rewrite this as

$$lE_1^3(D_1 + D_3/E_1)(1 + D_3/E_1(D_1 + D_3/E_1 - 1) - \frac{D_4}{2}((D_1 + D_3/E_1) - \sqrt{(D_1 + D_3/E_1)^2 + 4/l})).$$

We rewrite the third component of this product as

$$\begin{aligned}
&1 + (D_1 + D_3/E_1)(D_3/E_1 - D_4/2) - D_3/E_1 + \frac{D_4}{2}\sqrt{(D_1 + D_3/E_1)^2 + 4/l} \\
= \; &1 + 1/2(D_1 + D_3/E_1)(1 + D_3/E_1) - D_3/E_1 + \frac{D_4}{2}\sqrt{(D_1 + D_3/E_1)^2 + 4/l})
\end{aligned}$$

Since $E_1^3 < 0$ and we want to show that $q_2 < 0$, we need to get $1 + 1/2(D_1 + D_3/E_1)(1 + D_3/E_1) - D_3/E_1 + \frac{D_4}{2}\sqrt{(D_1 + D_3/E_1)^2 + 4/l} > 0$. Using the formula $1 - D_3/E_1 = -D_4$ we get an equivalent formulation

$$H := \frac{1}{2}(D_1 + D_3/E_1)(1 + D_3/E_1) + D_4(-1 + \frac{1}{2}\sqrt{(D_1 + D_3/E_1)^2 + 4/l}) > 0. \tag{38}$$

A short calculation verifies that since $l \geq 1$, $\frac{1}{2}\sqrt{(D_1 + D_3/E_1)^2 + 4/l} - 1 < (D_1 + D_3/E_1)/2$. Since $D_4 = -1 + D_3/E_1 < 0$ we get that

$$\begin{aligned}
H &> \frac{1}{2}(D_1 + D_3/E_1)(1 + D_3/E_1) + 1/2(-1 + D_3/E_1)(D_1 + D_3/E_1) \\
&= (D_1 + D_3/E_1)D_3/E_1 > 0.
\end{aligned}$$

Since $E_1^3 < 0$ this shows that $q_2 < 0$. By (32) we get $sgn(r_2^l) = sgn(r_4^l)$ for all l.

Finally, we compute the second component a_2^l of vector a. Setting $a_4^l = 1$ we get $a_2^l = n_2/d$ where n_2 can be written as

$$lE_1^3(D_1^2D_3/E_1 + D_1 - D_1D_3/E_1 + 2D_1D_3^2/E_1^2 + D_3/E_1 - D_3^2/E_1^2 + D_3^3/E_1^3) - E_1^3 + D_3E_1^2$$
$$-\frac{1}{2}(D_3 - E_1)E_1^2(l(D_1 + D_3/E_1)^2 + 2 + l(D_1 + D_3/E_1)\sqrt{(D_1 + D_3/E_1)^2 + 4/l}).$$

Observe that n_2 differs from (37) only in the sign in front of the square root term. The inequality analogous to to inequality (38) for n_2 is

$$\frac{1}{2}(D_1 + D_3/E_1)(1 + D_3/E_1) + D_4(-1 - \frac{1}{2}\sqrt{(D_1 + D_3/E_1)^2 + 4/l}) > 0.$$

Since $D_4 < 0$ this is satisfied and thus $n_2 < 0$. By (32) $sgn(a_2^l) = sgn(a_4^l)$ for all l . □

Corollary 4.3 *No map $\mathbf{F}_l$ with $l \geq 2$ has fixed points in its admissible domain Z_l. The map $\mathbf{F}_1$ has unique fixed point in Z_1 which is locally attracting.*

Proof. It follows from the Lemma 4.2 that the only fixed point of the maps $\mathbf{F}_l$ for all l, which is in the domain $(0, -, +, -)$, is the attractor of the map $\mathbf{F}_1$. This fixed point corresponds to the eigenvector of M_2M_1 with eigenvalue $\gamma_1 + \beta_1$.

We show that this attractor is always in the admissible domain Z_1 of the map $\mathbf{F}_1$. This domain is the subset of the domain $(0, -, +, -)$ characterized by the value of the function α. Values of this function on the domain are between 0 and 1. Computing $\alpha([0, a_2^1, a_3^1, a_4^1])$ using MAPLE, we get $\alpha([0, a_2^1, a_3^1, a_4^1]) = \frac{A}{B}$ where A can be written as $A = E_1^2(D_1 + D_3/E_1)(D_1 - D_3/E_1 + \sqrt{(D_1 + D_3/E_1)^2 + 4})$ and B is $B = E_1^2(D_1 + D_3/E_1)((1 + D_1)(D_1 + D_3/E_1 + \sqrt{(D_1 + D_3/E_1)^2 + 4}) - 2D_4)$. We observe that by (34) $A > 0$. Since $D_4 < 0$ and all other terms in B are positive, $B > 0$. It follows that $\alpha([0, a_2, a_3, a_4]) > 0$. Simplifying the ratio $\frac{A}{B}$ and estimating we get

$$\begin{aligned}
\alpha([0, a_2, a_3, a_4]) &= \frac{D_1 - D_3/E_1 + \sqrt{(D_1 + D_3/E_1)^2 + 4}}{(1 + D_1)(D_1 + D_3/E_1 + \sqrt{(D_1 + D_3/E_1)^2 + 4}) - 2D_4} \\
&< \frac{D_1 + D_3/E_1 + \sqrt{(D_1 + D_3/E_1)^2 + 4}}{(1 + D_1)(D_1 - D_3/E_1 + \sqrt{(D_1 + D_3/E_1)^2 + 4}} = \frac{1}{1 + D_1} < 1.
\end{aligned}$$

Thus $\alpha([0, a_2, a_3, a_4]) < 1$ and thus the attractor of $\mathbf{F}_1$ is always in the admissible domain of Z_1. □

Lemma 4.4 *Assume* **(H1)**,**(H2)** *and* **(H3)** *and (30). Any finite composition of matrices*

$$\mathcal{F} := F_{l_1} \circ F_{l_2} \dots F_{l_3}$$

has eigenvalue-eigenvector pairs $[0; (1, -E_1, 1, -E_1)]$; $[1; (0, 0, 1, -E_1)]$. Furthermore, the two remaining eigenvalues have the form $\gamma + \beta > 1$ and $\gamma - \beta < 1$.

Proof. Since every matrix F_l has eigenvalue-eigenvector pairs $[0; (1, -E_1, 1, -E_1)]$ and $[1; (0, 0, 1, -E_1)]$, their composition will have them also. Consider the reduced matrix of F_l

$$\left[\begin{array}{lll}
1 & l(E_1D_1 + D_3) & l(D_1 + D_3/E_1) \\
(1 + D_1)/E_1 & l(D_1(D_1 + D_3/E_1) + D_3/E_1) - l + 1 & lD_2(D_1 + D_3/E_1 - 1) \\
-1 + D_3/E_1 & l(D_3(D_1 + D_3/E_1) - (D_3 - E_1)) & 1 + U_{33}
\end{array}\right],$$

where $U_{33} := l(D_3/E_1(D_1 + D_3/E_1) - D_3/E_1) + l$. Recall that the reduced matrix is obtained from matrix F_l by dropping first column and first row. The reduced matrix has the same eigenvectors and eigenvalues except for the eigenvalue 0. We observe that the reduced matrix has the form

$$M = \left[\begin{array}{lll}
1 + U_{11} & U_{12} & U_{13} \\
U_{21} & 1 + U_{22} & U_{23} \\
U_{31} & U_{32} & 1 + U_{33}
\end{array}\right];$$

where all U_{ij} depend on l and other parameters ($U_{11} = 0$). The key observation is that a reduced matrix of a composition of maps $\mathcal{F}$ is the matrix of the same type as M. One can verify this by multiplying two matrices of the type M to get again a matrix of type M.

We now compute the characteristic polynomial of such a composition by computing characteristic polynomial of M. Since this is a reduced matrix, and 1 is an eigenvalue, the characteristic polynomial has the form

$$(x-1)(x^2 + V_1 x + V_2). \tag{39}$$

It is easy to see that $V_2 = det(M)$. Observe that since determinant of reduced form of M_1 and M_2 is 1, also the determinant of the reduced form of F_l is 1 for all l. This implies $V_2 = 1$.

Since $-trace\ (M)$ is the coefficient in front of x^2 in the characteristic polynomial we get by multiplying out (39) that $trace(M) = -V_1 + 1$. The trace of M has the form $3 + A$ where $A = U_{11} + U_{22} + U_{33}$. Thus $V_1 = -2 - A$ and the characteristic polynomial has the form

$$(x-1)(x^2 - x(A+2) + 1).$$

Computing the roots of the quadratic polynomial we get

$$\gamma \pm \beta := 1 + A/2 \pm \sqrt{(A/2)^2 + A}.$$

The result follows. □

Lemma 4.5 *Assume* **(H1)***),*(**H2)** *and* **(H3)** *and (30). Consider an arbitrary composition*

$$\mathbf{F}_{l_s}^{k_s} \dots \mathbf{F}_{l_1}^{k_1} \tag{40}$$

where at least one l_i is not equal to 1. *Then such a composition has no fixed points in its admissible domain Y for all such l_i and k_i.*

Proof. Without loss of generality we assume that $l_s \neq 1$. We compute the index along the boundary of admissible domain Y of $\mathbf{F}_{l_s}^{k_s} \dots \mathbf{F}_{l_1}^{k_1}$. It is clear that the index of $\mathbf{F}_{l_s}^{k_s} \dots \mathbf{F}_{l_1}^{k_1}$ along the boundary of Y is the index of $\mathbf{F}_{l_s}^{k_s}$ on the boundary of $\mathbf{F}_{l_{s-1}}^{k_{s-1}} \dots \mathbf{F}_{l_1}^{k_1}(Y)$. Observe that since $l_s \neq 1$, the map $\mathbf{F}_{l_s}^{k_s}$ does not have any fixed points in the region Z_{l_s} by Corollary 4.3. By the definition of Y

$$\mathbf{F}_{l_{s-1}}^{k_{s-1}} \dots \mathbf{F}_{l_1}^{k_1}(Y) \subset Z_{l_s}.$$

Since $F_{l_s}^{k_s}$ has no fixed points in Z_{l_s} the index of $\mathbf{F}_{l_s}^{k_s} \dots \mathbf{F}_{l_1}^{k_1}$ along the boundary of Y is zero.

By Lemma 4.4, the composition has an attractor, a repeller and the saddle at $(0, 0, 1, -E_1)$. Observe that the saddle is not in C_3 and the indices of the attractor and the repeller are both 1. Thus the index zero along the boundary of Y implies that there are no fixed points of $\mathbf{F}_{l_s}^{k_s} \dots \mathbf{F}_{l_1}^{k_1}$ in the admissible domain Y. □

Proof of Theorem 4.1 The first part follows from Theorem 3.1.

We show that for any $x \in Z_l$, $1 < l \leq k$ we have that the trajectory of x leaves Z_l after finitely many iterates and never comes back. Here a trajectory is a trajectory under a collection $\{\mathbf{F}_l\}_{l=1}^k$; if a points lands in Z_l the corresponding map $\mathbf{F}_l$ is applied. Assume to the contrary that there is a point $x \in Z_n$, $n > 1$, such that its trajectory visits Z_n infinitely many times. Denote the points of return by $\{y_i\}_{i=1}^\infty \in Z_n$. Since Z_n is compact there is a subsequence which converges to $z \in Z_n$. Observe that z must be a periodic trajectory under a composition of the type (40). Since $z \in Z_n$ one or the maps in composition is $\mathbf{F}_n$. This contradicts Lemma 4.5 in case $z \in Int(Z_n)$. For the case $z \in \partial Z_n$ we note that we could prove Lemma 4.5 for a slightly larger sets V_l, which contain Z_l in the interior, but do not contain any additional fixed points of $\mathbf{F}_l$, not already in Z_l. Then $z \in \partial Z_n \subset Int(V_l)$ and Lemma 4.5 applies with V_l instead of Z_l.

Hence all points in Z_l with $l > 1$ eventually leave Z_l. By necessity they must enter Z_1. The only fixed point in Z_1 is the attractor of $\mathbf{F}_1$, which correspond to a $(1, 1)$ periodic orbit in the R-flow. □

E-mail address: gedeon@math.montana.edu

References

[1] R. Edwards, Analysis of continuous-time switching networks, *Physica D*, 146, (2000), 165-199.

[2] R. Edwards, Chaos inneural and gene networks with hard switching, *Diff. Eq. Dyn. Sys.* vol. 9, No.3-4, (2001), 187-220.

[3] A.F. Filipov, *Differential Equations with discontinuous right hand sides*, Kluwer Academic Publishers 1988.

[4] T. Gedeon, Global dynamics of neural nets with infinite gain, *Physica D* 146, (2000), 200-212.

[5] L. Glass, Combinatorial and topological methods in nonlinear chemical kinetics, *J. of Chem. Physics*, vol. 63, No. 4, (1975), 1325-1335.

[6] L. Glass and J. Pasternack, Stable oscillations in mathematical models of biological control systems, *J. Math. Biol.*, **6**, (1978), 207-223.

[7] S. Grossberg, Contour enhancement, short term memory, and constancies in reverberating neural networks, *Stidies in Applied MAth.*, 52, (1973), 217-257.

[8] S. Grossberg, Competition, decision, and consensus, *J. Math. Anal. Appl.*, 66, (1978), 470-493.

[9] J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci.* vol. 79,pp. 2554-2558,(1982).

[10] J. J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, *Proc. Natl. Acad. Sci.* vol.81, pp.3088-3092, (1984).

[11] T. Mestl, Ch. Lemay and L. Glass, Chaos in high-dimensional neural and gene networks, *Physica D* 98, (1996), 33-52.

[12] J. W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics and Physics, 1997.

[13] T.W. Troyer, Feedforward Hebbian learning with nonlinear output units: a Lyapunov approach, *Neural Networks*, vol.9, No. 2, pp.321-328.