Chaotic solutions in slowly varying perturbations of Hamiltonian systems with applications to shallow water sloshing*

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Abstract

We study a slowly varying planar Hamiltonian system modeling shallow water sloshing. Using the Conley index theory for fast-slow systems of ODEs, we prove the existence of complicated dynamics in the system which is described in terms of symbolic sequences of integers. This includes the solutions proven by Hastings and McLeod as well as those conjectured by them.

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1 Introduction

Slowly varying Hamiltonian systems model various kinds of physical phenomena, which sometimes exhibit very complicated behaviors (e.g. [16, 3, 4, 2]). The purpose of this paper is to show that such systems can have a rich variety of solutions whose behaviors can be described in terms of symbolic coding.

The equation we consider is a slowly varying planar Hamiltonian system possibly with a higher order perturbation given as follows:

$$\frac{du}{dt} = J\nabla H(u, \varepsilon t) + \varepsilon^2 h(u, \varepsilon t, \varepsilon), \quad u \in \mathbb{R}^2, \tag{1.1}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic structure on the plane and $\varepsilon > 0$ is taken to be sufficiently small. The Hamiltonian function H is assumed to be C^3 and the perturbation term h is C^2. One can rewrite this system in the form of a multi-time scale vector field on $\mathbb{R}^3$ as follows:

$$\dot{u} = J\nabla H(u, \lambda) + \varepsilon^2 h(u, \lambda, \varepsilon), \tag{1.2}$$

$$\dot{\lambda} = \varepsilon. \tag{1.3}$$

For the study of chaotic dynamics for time-dependent Hamiltonian systems, one often assumes that the Hamiltonian is time periodic, thereby reducing it to the study of the Poincaré map associated to the time period. This approach is not possible here since the Hamiltonian is not necessarily time periodic; it merely has an oscillatory character which will be made precise later.

We make the following two hypotheses for the system (1.1):

Hypothesis

(H1) For $\varepsilon = 0$, the system (1.2) reduces to a one-parameter family of planar autonomous Hamiltonian systems

$$\frac{du}{dt} = J\nabla H(u, \lambda). \tag{1.4}$$

We suppose that, for each λ, the Hamiltonian system (1.4) has two equilibrium points $A(\lambda)$ and $B(\lambda)$ which depend C^1 in λ, that $A(\lambda)$ is elliptic, $B(\lambda)$ is hyperbolic with a homoclinic orbit enclosing the other equilibrium point $A(\lambda)$ in the u-plane.

We let $\mathcal{A} = \{A(\lambda)\}$ and $\mathcal{B} = \{B(\lambda)\}$, respectively. Notice that the curve $\mathcal{B}$ is a normally hyperbolic invariant manifold whose stable and unstable manifolds close up and form a surface of homoclinic orbits.

(H2) Let $S(\lambda)$ denote the area bounded by the homoclinic orbit in the u-plane for each λ. We assume that the function $S(\lambda)$ is C^1 and its derivative $S'(\lambda)$ is *oscillating*, in the sense that there exists a finite or infinite sequence of disjoint closed intervals $\Lambda_i = [\lambda_i^-, \lambda_i^+]$ such that $\lambda_i^- < \lambda_i^+ < \lambda_{i+1}^-$, $S'(\lambda_{2i}^-) > 0,\ S'(\lambda_{2i}^+) < 0$

and $S'(\lambda_{2i+1}^-) < 0$, $S'(\lambda_{2i+1}^+) > 0$ for all i. Moreover, we assume that $\lambda_{i+1}^- - \lambda_i^+$, the gap between the adjacent intervals is uniformly bounded away from zero.

Notice that if $H(u, \lambda)$ is periodic in λ and if $S(\lambda)$ is not a constant function, then $S'(\lambda)$ is always oscillating with an infinite sequence of such intervals.

Definition 1.1 A solution $X(t) = (u(t), \lambda(t))$ of equations (1.2)-(1.3) is said to *oscillate k times* over an interval $\Lambda = [\lambda^-, \lambda^+]$, if the oscillation number of the solution with respect to the curve $\mathcal{A}$ over Λ is equal to k. Here the oscillation number is defined as the homotopy class of the following closed loop

$$l = \left(\bigcup_{\lambda(t) \in \Lambda} X(t) \cup B(\lambda(t)) \right) \bigcup \left(\bigcup_{\lambda(t) \in \partial\Lambda} \overline{X(t)B(\lambda(t))} \right)$$

in the fundamental group of $\mathbb{R}^3 \setminus \mathcal{A}$ which is isomorphic to $\mathbb{Z}$, where $\overline{PQ}$ stands for the line segment connecting the points P and Q.

Let $\Lambda = [\lambda^-, \lambda^+]$ be an interval such that $S'(\lambda^+)$ and $S'(\lambda^-)$ have opposite signs. Applying the results of [7] one obtains two sets of non-negative integers, $\Sigma_\Lambda(\varepsilon)$ and $\Theta_\Lambda(\varepsilon)$, which consist of realizable oscillation numbers over the interval Λ associated with the slow flow speed $\varepsilon > 0$ in the following sense:

- If $S'(\lambda^-) < 0$ and $S'(\lambda^+) > 0$, then for any $k \in \Sigma_\Lambda(\varepsilon)$ there exists a solution of (1.2)-(1.3) that oscillates k times over Λ.
- If $S'(\lambda^-) > 0$ and $S'(\lambda^+) < 0$, then for any $k \in \Theta_\Lambda(\varepsilon)$ there exists a solution of (1.2)-(1.3) that oscillates k times over Λ.

While the precise definition of these sets will be given in Section 3 they have several important properties:

- for all $\varepsilon > 0$, $0 \in \Sigma_\Lambda(\varepsilon)$ and $0 \in \Theta_\Lambda(\varepsilon)$;
- for ε sufficiently small, $\Sigma_\Lambda(\varepsilon)$ contains at least two elements;
- by choosing ε small enough, $\Theta_\Lambda(\varepsilon)$ can be made to have arbitrarily many elements.

Now we are ready to state the main theorem of this paper.

Main Theorem *Assume the hypotheses (H1) and (H2) are satisfied, and hence there exists a sequence of disjoint closed intervals $\{\Lambda_i\}$ as in (H2) as well as the integer sets $\Sigma_{\Lambda_i}(\varepsilon)$ and $\Theta_{\Lambda_i}(\varepsilon)$. Then, for any sequence of integers $\{\sigma_i\}$ with*

$$\sigma_{2i} \in \Theta_{\Lambda_{2i}}(\varepsilon), \quad \textit{and} \quad \sigma_{2i+1} \in \Sigma_{\Lambda_{2i+1}}(\varepsilon),$$

there exists a solution of (1.2)-(1.3) which oscillates σ_i times over Λ_i.

This theorem asserts that under these hypotheses, there exists a set of solutions which have a certain number of oscillations over the given intervals Λ_i. Furthermore, these oscillations are prescribed in terms of symbolic sequences of integers. Therefore, if an infinite number of the sets $\Theta_{\Lambda_{2i}}(\varepsilon)$ and $\Sigma_{\Lambda_{2i+1}}(\varepsilon)$ contain more than one element, then there is a non-trivial symbolic coding of the sequence of oscillations and hence the set of oscillating solutions is very complicated or may be said chaotic. More precisely, the symbolic coding shows that there are uncountably infinitely many topologically distinct bounded orbits. The following corollary gives sufficient conditions for these sets to contain more than one element.

Corollary 1.2 *If the Hamiltonian $H(u, \lambda)$ is periodic in λ, then there exists a choice of infinitely many intervals Λ_i with the following property: for any given positive integer K there exists $\bar{\varepsilon} > 0$ such that for any ε with $0 < \varepsilon < \bar{\varepsilon}$, we have*

$$\#\Sigma_{\Lambda_{2i+1}}(\varepsilon) \geq 2, \quad \textit{and} \quad \#\Theta_{\Lambda_{2i}}(\varepsilon) \geq K,$$

for all i, where $\#X$ stands for the cardinality of the set X.

Remark 1.3 As will become clear from the proof, the periodicity is not necessary to obtain the conclusion of the above corollary. The sets $\Sigma_{\Lambda_i}(\varepsilon)$ and $\Theta_{\Lambda_i}(\varepsilon)$ are completely determined by $H(u, \lambda)$ restricted to Λ_i and by the assumption that $\lambda_{i+1}^- - \lambda_i^+$ are uniformly bounded away from zero. Furthermore, as was mentioned earlier, for each Λ_i there exists ε_i such that $\#\Sigma_{\Lambda_i}(\varepsilon) \geq 2$ and $\#\Theta_{\Lambda_i}(\varepsilon)$ being arbitrarily large for $0 < \varepsilon < \varepsilon_i$. Thus, to obtain an uncountable number of topologically distinct orbits, it is sufficient to assume that there are infinitely many ε_i uniformly bounded away from zero.

Corollary 1.4 *For any finite collection of intervals Λ_i and for any given positive integer K, there exists $\bar{\varepsilon} > 0$ such that for any ε with $0 < \varepsilon < \bar{\varepsilon}$, we have*

$$\#\Sigma_{\Lambda_{2i+1}}(\varepsilon) \geq 2, \quad \textit{and} \quad \#\Theta_{\Lambda_{2i}}(\varepsilon) \geq K,$$

for all i.

This work was originally motivated by a result of Hastings and McLeod [3], in which they studied complicated dynamics in a problem of shallow water sloshing. We will discuss this example in Section 2. The proof of the main theorem is given in Section 3. The proof uses the Conley index theory, in particular, a recently developed method adapted for singularly perturbed ODEs, which is summarized in Appendix A. Section 4 is devoted to discussions on related results and concluding remarks.

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2 Example: shallow water sloshing

H. Ockendon, J. R. Ockendon and A. D. Johnson [11] studied the problem of shallow water sloshing. This is a two-dimensional irrotational fluid motion in a rectangular tank of inviscid fluid with horizontal oscillatory forcing. Assuming that forcing is time periodic, they obtained a partial differential equation model for this problem. They also derived, as a first order approximation of the PDE model, a fourth order ordinary differential equation:

$$\frac{1}{3}\kappa^2 f''''(t) - (\rho - \frac{1}{3}\kappa^2)f''(t) - 3f'(t)f''(t) = \frac{2}{\pi}\sin t, \tag{2.1}$$

where the leading term of the velocity potential is given as $f(t-x) + f(t+x)$. This approximation is valid when the amplitude of the oscillation is very small with respect to the horizontal length of the rectangular tank. The constants κ and ρ are determined from the size of the tank as well as the amplitude of the forcing. We are interested in the case the constant κ is small. Note that in the papers of [11] and [3] the constant λ is used in place of ρ above. We used different notation in order to avoid confusion, since throughout this paper λ represents the slow variable.

S. P. Hastings and J. B. McLeod [3] studied this ODE model when the constant $\rho = \frac{1}{3}\kappa^2$. In this case one can integrate the equation once, and by putting $g = f'$, reduce the ODE to the following second order equation:

$$\frac{1}{3}\kappa^2 g''(t) = \frac{3}{2}g(t)^2 - \frac{2}{\pi}\cos t + C,$$

where C is an integration constant, or equivalently, it can be written as a fast-slow system of the form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 - 1 - c - \cos(\varepsilon t) \end{aligned}$$

after appropriate rescaling of variables. Notice that this is a slowly varying time-periodic planar Hamiltonian system, and if $c > 0$, it satisfies the hypothesis (H1) given in Section 1.

Using what they call "simple shooting method", Hastings and McLeod noticed that any solution $x(t)$ of this equation cannot have a minimum value in the range $-\sqrt{c} < x < \sqrt{c}$, and hence relative minima of the solution are sharply distinguished as positive (larger than $\sqrt{c}$) or negative (less than $-\sqrt{c}$). This allows them to introduce a symbolic coding of solutions in terms of the number of successive positive minima and negative minima. Using this idea, they have proved, among other things, the existence of solutions which are coded by arbitrary sequence of symbols 1's (negative minima) and 0's (positive minima). Moreover these solutions have minima corresponding to the symbols in each 2π-interval of $\lambda = \varepsilon t$. "Subharmonic" solutions which have more negative minima with one or no positive minima in each 2π-interval of λ are also proven to exist.

Numerically, Hastings and McLeod have observed the existence of more complicated solutions whose existence is not proven in [3]. In fact, they conclude the paper by the following remark:

> .... It seems likely that there are infinite families of subharmonics with n spikes near even multiples of π and zero or one spike near odd multiples of π, ...

Using Corollary 1.2, one can in fact prove the existence of such families of solutions. In order to apply it to the system, one has to verify another hypothesis (H2). From a simple calculation, the area function of the above planar Hamiltonian system is given by

$$S(\lambda) = \frac{24\sqrt{2}}{5}(1 + c + \cos\lambda)^{3/4}.$$

Clearly this is not a constant function and its derivative $S'(\lambda)$ is periodic and hence infinitely oscillating. Observe that the zeroes of $S'(\lambda)$ are multiples of π, and hence one can take the intervals Λ_{2i} as a compact neighborhood of an even multiple of π, whereas Λ_{2i+1} a compact neighborhood of an odd multiple of π. Therefore, from Corollary 1.2, we conclude that, given an arbitrary sequence of integers $\{\sigma_i\}$ with the property that $\sigma_{2i} \in \Theta_{\Lambda_{2i}}(\varepsilon)$ and $\sigma_{2i+1} \in \Sigma_{\Lambda_{2i+1}}(\varepsilon)$, there exists a solution which oscillates σ_i times in a neighborhood of an integer multiple of π in a prescribed manner. This assertion is almost exactly what is conjectured at the end of [3]; the difference is that "zero or one spike" assertion must be replaced by "zero or non-zero spike". However, if we use a slightly more precise information about the derivative of the area function $S'(\lambda)$, namely it is strictly monotone over the chosen intervals Λ_i so that the set $\Sigma_i(\varepsilon)$ always contain 0 and 1 for sufficiently small ε, then we can exactly prove the conjectural statement of Hastings and McLeod.

The same analysis can be applied to the original equation (2.1) derived by [11]. Integrating (2.1) once and rewriting it as in the form of the system

$$\begin{aligned} \dot{y} &= z, \\ \dot{z} &= \frac{3}{2}y^2 + (\rho - \varepsilon^2)y - c - \frac{2}{\pi}\cos(\varepsilon t), \end{aligned}$$

where $y(t) = f'(\varepsilon t)$ and ε is a small constant proportional to κ, one again has a family of slowly varying planar Hamiltonian systems which satisfies the assumptions (H1) and (H2). The Hamiltonian function for this system is given by

$$H(y, z, \lambda) = \frac{1}{2}z^2 - \frac{1}{2}y^3 - \frac{1}{2}\rho y^2 + cy + \frac{2}{\pi}y\cos\lambda$$

for each fixed λ. Straightforward calculation shows that the area function $S(\lambda)$ is given by

$$S(\lambda) = \frac{8}{15}\left\{\rho^2 + 6\left(c + \frac{2}{\pi}\cos\lambda\right)\right\}^{5/4}.$$

The derivative $S'(\lambda)$ has a zero whenever λ is an integer multiple of π, it is decreasing if λ is an odd multiple of π, and increasing if λ is an even multiple of π. Therefore, we basically have the same conclusion about the existence of solutions that behave according to symbolic sequences $\{\sigma_i\}$ as before.

Hastings and McLeod studied similar problems in [4] and [2], to which the results of this paper are equally applicable. We will discuss more about the difference between our approach and other approaches in Section 4.

3 Proof of Main Theorem

3.1 Splitting of homoclinic surfaces

Consider

$$\begin{aligned}\dot{u} &= J\nabla H(u,\lambda) + \varepsilon^2 h(u,\lambda,\varepsilon)\\ \dot{\lambda} &= \varepsilon\end{aligned}$$

As assumed in (H1), this system has a surface of homoclinic orbits when $\varepsilon = 0$. Since the curve $\mathcal{B}$ consisting of equilibrium points $B(\lambda)$ for all $\lambda \in \mathbb{R}$ is normally hyperbolic, the stable and unstable manifolds $W^s(\mathcal{B})$ and $W^u(\mathcal{B})$ persist for non-zero but small ε. One can then see how these manifolds intersect by measuring the splitting distance as follows. Take a transverse cross section to the surface of homoclinic orbits for $\varepsilon = 0$. This continues to be a cross section to $W^s(\mathcal{B})$ and $W^u(\mathcal{B})$ if $\varepsilon > 0$ is small enough. Let $d(\lambda,\varepsilon)$ be the signed distance measured from $W^u(\mathcal{B})$ to $W^s(\mathcal{B})$ in the intersection of the cross section and the plane given by each λ.

Theorem 3.1 ([13, 16, 12, 15]) *The signed distance $d(\lambda,\varepsilon)$ of $W^u(\mathcal{B})$ and $W^s(\mathcal{B})$ defined above satisfies the following asymptotic expression:*

$$\frac{\partial d}{\partial \varepsilon}(\lambda, 0) = \text{constant} \cdot S'(\lambda),$$

where $S(\lambda)$ is the area enclosed by the homoclinic orbit of the planar Hamiltonian system at λ when $\varepsilon = 0$. In particular, if $S'(\lambda)$ takes non-zero values with different sign at λ^+ and λ^-, then, for sufficiently small $\varepsilon > 0$, $W^u(\mathcal{B})$ and $W^s(\mathcal{B})$ intersect topologically transversely over the interval between λ^- and λ^+.

Note that the formula

$$d(\lambda,\varepsilon) \approx \varepsilon(S'(\lambda) + O(\varepsilon)) \tag{3.1}$$

is true but the constants depend on the choice of cross sections where the distance is measured. In particular, the term $O(\varepsilon)$ is proportional to some constant which depends on the location of the cross section, and therefore, if the cross section is taken too close to the hyperbolic saddle $B(\lambda)$, the constant may be very large so that the second term dominates the first term for small but non-zero ε. Even in such a case, the formula remains true if one chooses a smaller value of ε.

This is a version of Melnikov-type theorems. The idea of Melnikov integrals was applied to slowly varying ODEs by Robinson [13], and then further developed in [16], [15], [12], and others. This particular formulation as well as its proof can be found in [12].

3.2 Existence of infinitely many connecting orbits

Suppose there exist $\lambda^- < \lambda^+$ such that the function $S'(\lambda)$ takes non-zero values with different sign at $\lambda = \lambda^\pm$. Let $\Lambda = [\lambda^-, \lambda^+]$ and let $g(\lambda)$ be a smooth function which vanishes at $\lambda^\pm$ with $g'(\lambda^-) > 0$ and $g'(\lambda^+) < 0$ and which does not vanish elsewhere in Λ.

Consider the following modified equation:

$$\begin{aligned} \dot{u} &= J\nabla H(u,\lambda) + \varepsilon^2 h(u,\lambda,\varepsilon) - \delta S'(\lambda)\nabla H(u,\lambda), \\ \dot{\lambda} &= \varepsilon g(\lambda). \end{aligned} \tag{3.2}$$

Note that, from the above conditions of g, the points $B(\lambda^\pm)$ on $\mathcal{B}$ are hyperbolic equilibria. The term $\delta S'(\lambda)\nabla H(u,\lambda)$ is the artificial dissipation; this is introduced in order to make the splitting distance of $W^u(\mathcal{B})$ and $W^s(\mathcal{B})$ at $\lambda = \lambda^\pm$ even bigger so that they remain separated when $\varepsilon = 0$ but $\delta > 0$.

We are interested in finding orbits connecting the equilibria $B(\lambda^\pm)$ in $\mathcal{B}$ at $\lambda = \lambda^\pm$. Since we only look for topologically distinguishable connecting orbits, we will measure the oscillation number with respect to the curve $\mathcal{A}$ as defined in §1. Below, we only consider $S'(\lambda^-) > 0$ and $S'(\lambda^+) < 0$ but both $\varepsilon > 0$ and $\varepsilon < 0$. It is easy to see that the $\varepsilon < 0$ case is equivalent to $S'(\lambda^-) < 0$ and $S'(\lambda^+) > 0$ with $\varepsilon > 0$.

Theorem 3.2 ([7]) *(1) For sufficiently small $\varepsilon > 0$, there exist at least 2 but at most finitely many sets of connecting orbits with distinct oscillation numbers.*

(2) For sufficiently small $\varepsilon < 0$, there exist infinitely many connecting orbits with distinct oscillation numbers.

The proof of this theorem goes as follows. First we take an isolating neighborhood which is homeomorphic to a 3-disk and which contains the whole surface of homoclinic orbits over the interval Λ, and remove a tubular neighborhood of the curve $\mathcal{A}$. This is homeomorphic to a fattened cylinder, or the product of an annulus and an interval, whose universal covering space is homeomorphic to a 3-disk. In this covering space, there are infinitely many lifts of the piece of the curve $\mathcal{B}$ over the interval Λ. Let $\mathcal{B}_i$ be those lifts. For each lift $\mathcal{B}_i$, there are also the corresponding lifts $B_i^\pm$ of the equilibria $B(\lambda^\pm)$. From the construction, the flow lifted on the covering space is monotone in the sense that an orbit passing near some $\mathcal{B}_i$ may pass near $\mathcal{B}_j$ with $j > i$, but never do so with $j < i$. This comes from the assumption that the orbits in the base space rotates in one direction with respect to the curve $\mathcal{A}$. Therefore, we look for connecting orbits from B_i^- to B_j^+ with $j \geq i$ for $\varepsilon > 0$, and those from B_i^+ to B_j^- for $\varepsilon < 0$.

The difference between the two assertions of Theorem 3.2 for positive and negative ε is completely topological. Intuitively, readers can easily be convinced by the figures in [7]. Noticing this, in order to find those connecting orbits, we use the Conley index theory, and in particular the theory of transition matrices. The transition matrix is a matrix whose non-zero (i,j)-entry shows the existence of the connecting orbit from $B_i^\pm$ to $B_j^\mp$. The essential part of the proof is very simple: one computes the transition matrix T_- of the covering space for $\varepsilon < 0$, using the information of the flow on the base space. If the (i,j)-entry of the matrix T_- is non-zero, then we can conclude that there exists a connecting orbit which if projected down to the base space has a oscillation number $j - i$. It turns out that only finitely many oscillation numbers are realized by this matrix. On the other hand, it is known that T_+, the transition matrix for $\varepsilon > 0$, is given by taking the inverse of T_-. This implies that in this case one must have infinitely many integers $j-i$ from non-zero entries of T_+. For the detail of the argument, see [7].

One of the important consequences of the proof is that each connecting orbit found by the above argument must lie in an isolating neighborhood, since it is detected by the Conley index theory. Moreover, the transition matrix is a linear map acting on the sum of the homology groups of the individual Morse components (the equilibria $B_i^{\pm}$ in our case) to itself, and therefore if a connecting orbit from $B_i^{\pm}$ to $B_j^{\mp}$ is detected, the transition matrix carries the generator of the homology Conley index of $B_i^{\pm}$ to that of $B_j^{\mp}$ non-trivially. It is this information we will make use of later in order to construct the symbolic coding. Notice that since this is purely topological information, it does not depend on the choice of the function g. Hence one can choose g so that it is identically equal to 1 on an open interval slightly smaller than $\Lambda = [\lambda^-, \lambda^+]$.

Now we give the precise definition of the sets $\Sigma_\Lambda(\varepsilon)$ and $\Theta_\Lambda(\varepsilon)$ introduced in the Introduction for our Main Theorem. Recall that we are considering the case $S'(\lambda^-) > 0$ and $S'(\lambda^+) < 0$. The other case can be treated in a similar way. From the argument in [7], we can obtain the transition matrix for the slow flow with small enough speed $-\varepsilon < 0$ and obtain the set Σ_Λ of finitely many non-negative integers which are oscillation numbers of the connecting orbits for the equation (3.2) with ε replaced by $-\varepsilon < 0$. Namely, these are the numbers $j-i$ for which the (i,j)-entry of the transition matrix T_- is 1. If the slow flow is of constant speed $-\varepsilon < 0$, then not all oscillations may be realized by solutions over $[\lambda^-, \lambda^+]$, and hence we denote by $\Sigma_\Lambda(\varepsilon)$ the subset of Σ_Λ which consists of only the realizable oscillation numbers over Λ in the equation (3.2) with $g(\lambda) \equiv -1$. Notice that 0 always belongs to the set and hence $\Sigma_\Lambda(\varepsilon) \neq \emptyset$. Moreover, if ε is chosen small enough, $\Sigma_\Lambda(\varepsilon) = \Sigma_\Lambda$, since Σ_Λ is a finite set.

From the result of [7], the oscillation of connecting orbits in the slow flow with $\varepsilon > 0$ is given by inverting the transition matrix T_- for $\varepsilon < 0$. More precisely, the possible oscillation numbers are given as follows: Let $\theta(n)$ be the cardinality of

$$\{(j_1, \ldots, j_k) \mid j_l \in \Sigma_\Lambda \ (1 \leq l \leq k),\ \sum_{l=1}^{k} j_l = n\}.$$

Then the set Θ_Λ is given by

$$\Theta_\Lambda = \{n \mid \theta(n) \equiv 1 \bmod 2\}.$$

It is easy to see from the argument in [7] that this is in fact the set of all possible oscillation numbers of the connecting orbits of the equation (3.2) with $\varepsilon > 0$, since, if a number n belongs to the set Θ_Λ, it means that the $(i, i+n)$-entry of the inverse of the transition matrix T_- is 1 for any i. Again, if the slow flow is of constant speed $\varepsilon > 0$, namely, $g(\lambda) \equiv 1$ in the $\dot{\lambda}$-equation, then not all the above oscillation numbers in Θ_Λ may be realized, and the set $\Theta_\Lambda(\varepsilon)$ denotes the subset of Θ_Λ which consists of the realizable oscillation numbers over Λ in the equation (3.2) with $g(\lambda) \equiv 1$ and $\varepsilon > 0$.

3.3 Concatenation of connecting orbits

Let us go back to our original problem (1.2)-(1.3). Recall from the assumption (H2) that there are intervals $\Lambda_i = [\lambda_i^-, \lambda_i^+]$ such that $S'(\lambda_{2j}^-) > 0,\ S'(\lambda_{2j}^+) < 0,\ S'(\lambda_{2j+1}^-) < 0$

and $S'(\lambda_{2j+1}^+) > 0$ for any possible $i = 2j$ or $2j+1$. Those intervals are disjoint. The main part of the proof is to show that, given a finite sequence $\sigma^J = \{\sigma_{-J}, \ldots, \sigma_J\}$ of admissible integers, namely, the integers satisfying $\sigma_{2i-1} \in \Sigma_{\Lambda_{2i-1}}(\varepsilon)$ and $\sigma_{2i} \in \Theta_{\Lambda_{2i}}(\varepsilon)$, there exists an orbit which oscillates σ_i times over the intervals Λ_i for $i = -J, ..., J$.

Consider the equation

$$\begin{aligned} \dot{u} &= J\nabla H(u,\lambda) + \varepsilon^2 h(u,\lambda,\varepsilon) - \delta S'(\lambda)\nabla H(u,\lambda), \\ \dot{\lambda} &= \varepsilon \tilde{g}_J(\lambda), \end{aligned}$$

where $\tilde{g}_J(\lambda)$ vanishes at λ_{-J}^- and λ_J^+ with $g'(\lambda_{-J}^-) > 0$ and $g'(\lambda_J^+) < 0$, and $g = 1$ everywhere in a slightly smaller open interval of $[\lambda_{-J}^-, \lambda_J^+]$. Similarly to the previous subsection, we can lift this equation onto an appropriate covering space.

If one furthermore modifies the equation as in the previous subsection so that the points on the curve $\mathcal{B}$ corresponding to the boundary points of each Λ_i become equilibria, one can find a connecting orbit which oscillates σ_i times over Λ_i. Moreover, the connecting orbit carries a homology generator of the fast dynamics at λ_i^- to that at λ_i^+ in a non-trivial manner, and this correspondence of the generators is given by a non-zero entry of the transition matrix. Since the generators of the fast dynamics can be defined without creating virtual equilibria on $\mathcal{B}$ at $\lambda_i^\pm$, the homology information about the orbits that oscillate σ_i times over the interval Λ_i is retained.

This argument shows that for each Λ_i one can construct a σ_i-fold covering space in which the transition matrix has a non-zero entry corresponding to orbits oscillating σ_i times over Λ_i. This part of the covering space satisfies the definition of the box defined in the Appendix as well as a sufficiently small tubular neighborhood of a lift of the curve $\mathcal{B}$ which connects an exit part of the box over Λ_i and the entrance side of the next box Λ_{i+1} satisfies the tube also defined in Appendix. Therefore we have a collection of boxes and tubes which are compatible in the sense of Definition A.4. If we take an isolating neighborhood of the (artificially created) hyperbolic equilibrium points on $\mathcal{B}$ at λ_{-J}^- and at λ_J^+, respectively, they satisfy the condition of caps (see Definition A.4), one being a repelling cap $C(R)$ and another an attracting cap $C(A)$, and hence we have constructed a tube-box-cap collection.

We apply the following TBC collection theorem in Appendix and conclude the existence of a connecting orbit $(u_J(t), \lambda_J(t))$ that oscillates according to the finite symbolic sequence σ^J.

Theorem 3.3 ([1]) *(1) The union N_J consisting of the tubes, boxes and caps is an isolating neighborhood for sufficiently small $\varepsilon > 0$.*

(2) $(\mathrm{Inv}C(R), \mathrm{Inv}C(A))$ *is an attractor-repeller pair for* $\mathrm{Inv}N_J$.

(3) If the composition of the transition maps $T_J \circ \cdots \circ T_{-J} \neq 0$, then there exists a connecting orbit $(u_J(t), \lambda_J(t))$ from $\mathrm{Inv}C(R)$ *to* $\mathrm{Inv}C(A)$.

In our case, all T_i are the non-zero entries of the transition matrix on each Λ_i, and therefore we have proven that there exists a connecting orbit $(u_J(t), \lambda_J(t))$. It remains to remove the artificial dissipation term given by δ, which will be done by using the following lemma.

Key Lemma *If N_J is an isolating neighborhood for the connecting orbit $(u_J(t), \lambda_J(t))$ for some $\bar{\delta} > 0$, then it is an isolating neighborhood for all $\delta \in [0, \bar{\delta}]$.*

Proof. Suppose N_J fails to be an isolating neighborhood for some $\delta \geq 0$. Since the only invariant set inside N_J is either the hyperbolic equilibria in the caps or a connecting orbit between them, this supposition means that there exists a connecting orbit $(u_\delta(t), \lambda_\delta(t))$ which has an inner tangency at a point of the boundary of N_J.

The isolating neighborhood N_J was chosen in such a way that the homoclinic surface at $\varepsilon = 0, \delta = 0$ is $O(1)$ distance away in the u directions from the boundary of the box. Therefore the only way to lose isolation is at the boundary of the boxes corresponding to $\lambda =$ const. Let $\Lambda_i = [\lambda_i^-, \lambda_i^+]$ be the interval corresponding to the box in which the isolation is violated. We consider the case where the loss of isolation occurs at $\lambda = \lambda_i^-$. The other case is treated similarly.

From [14], the flow inside a fixed neighborhood of the normally hyperbolic manifold $\mathcal{B}$ is C^1-linearizable. Since the size of a tube is of $O(1)$ independent of ε, orbits staying entirely in a tube and its adjacent box are of $O(e^{-1/\varepsilon})$-close to the unstable manifold of the normally hyperbolic curve $\mathcal{B}$ when they leave the tube. More generally, if an orbit stays in a size $O(\varepsilon)$ fixed neighborhood of $\mathcal{B}$ for $O(1/\varepsilon)$ amount of time before leaving the neighborhood, the orbit will remain within $O(e^{-1/\varepsilon})$ close to the unstable manifold of $\mathcal{B}$ for $O(1)$ amount of time outside the neighborhood. In fact, under the C^1-linearizing coordinates, the flow in the neighborhood can simply be written as

$$\dot{u} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} u, \quad \dot{\lambda} = \varepsilon.$$

Then the standard Gronwall estimate shows the assertion. By the same reasoning applied to the time reversed flow, an orbit entering the neighborhood of $\mathcal{B}$ and staying there for $O(1/\varepsilon)$ amount of time must stay within $O(e^{-1/\varepsilon})$ close to the stable manifold of $\mathcal{B}$ for $O(1)$ amount of time right before it enters the neighborhood.

Notice that, inside the box over Λ_i, the connecting orbit can stay only $O(1)$ amount of time outside the neighborhood of $\mathcal{B}$, otherwise the orbit would make unbounded number of rotations around $\mathcal{A}$ and hence would have unbounded oscillation number. Therefore, the above estimate shows that the connecting orbit is within $O(e^{-1/\varepsilon})$ close to both stable and unstable manifolds of $\mathcal{B}$ outside the neighborhood of $\mathcal{B}$.

On the other hand, from (3.1), the splitting distance of $W^u(\mathcal{B})$ and $W^s(\mathcal{B})$ is at least of $O(\varepsilon)$ at the side boundary of the box over Λ_i. This leads to a contradiction, since the connecting orbit $(u_\delta(t), \lambda_\delta(t))$ loses isolation by making inner tangency at the intersection of the tube and box and leaves the neighborhood of $\mathcal{B}$ where it cannot be both $O(e^{-1/\varepsilon})$ close to both $W^u(\mathcal{B})$ and $W^s(\mathcal{B})$ because of (3.1). This completes the proof. □

The Key Lemma enables us to compute the Conley homology index of N_J with δ and $\varepsilon > 0$, which implies the existence of a connecting orbit in N_J even for δ.

3.4 Final step

Let $\sigma = \{\sigma_i\}_{i\in\mathbb{Z}}$ be an admissible infinite sequence, and let $\sigma^J = \{\sigma_{-J}, \ldots, \sigma_J\}$ be its finite truncation from σ_{-J} to σ_J. Let $(u_J(t), \lambda_J(t))$ be a connecting orbit which behaves accordingly to σ^J. Choosing a subsequence, if necessary, one obtains a convergent sequence $(u_J(t), \lambda_J(t))$ and its limit $(u(t), \lambda(t))$ is a desired solution.

This completes the proof of Main Theorem.

4 Discussion

Our main result shows the existence of the conjectured symbolic sequences of Hastings and McLeod [3]. Also it can be applied to other examples discussed in [2, 4].

Hastings and McLeod approach is based on a shooting method using information of solutions. Their method is elementary, but requires good guess for the expected structure of solutions. Therefore it may not be systematically applied to various examples.

A standard Melnikov method can also be applied to the sloshing problem. See a similar discussion in [6] about the Melnikov approach to slowly varying pendulum studied in [3] and a more recent result in [15]. However, in order to conclude the existence of complicated dynamics in terms of symbolic sequences, one may need transverse intersection of stable and unstable manifolds, as well as time periodicity. The latter is required in order to reduce the ODE problem to a problem for diffeomorphisms so that one can apply the Poincaré-Birkhoff-Smale theorem about the existence of horseshoes near a transeverse homoclinic point. Moreover, proving the existence of various types of oscillating solutions by using the Melnikov method is technically quite involved and requires a more sophisticated higher order Melnikov theory as developed in [15]. Our method does not need the transversality nor the periodicity. It can be applied to slowly varying systems with not necessarily time periodic (but merely oscillatory) forcing.

Finally we remark that Lerman and Shil'nikov [5] have developed a method that allows one to concatenate a finite or infinite sequence of homoclinic/heteroclinic solutions geometrically. This method seems to be applicable to the sloshing problem and lead to the same conclusion as ours, once certain hypotheses such as transversality of the homoclinic/heteroclinic solutions are verified. Moreover, Lerman and Shil'nikov method may be simpler than the Melnikov theory in order to obtain the conclusion. Again, our method need less conditions than [5], and hence it can potentially be applied to broader class of problems.

A Conley index theory for fast-slow systems

Consider the family of differential equations on $\mathbb{R}^n \times \mathbb{R}$ given by

$$\begin{aligned} \dot{x} &= f(x, \lambda, \varepsilon) \\ \dot{\lambda} &= \varepsilon g(x, \lambda, \varepsilon) \end{aligned} \tag{A.1}$$

where $f(x, \lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g(x, \lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are C^1 functions and $\varepsilon \geq 0$. In this paper, we only consider the case where $g(x, \lambda) > 0$. A more general case is treated in [1]. The solutions to this equation generate a flow

$$\varphi^\varepsilon : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}.$$

In the special case $\varepsilon = 0$, (A.1) has a simpler form since λ is a constant. We can view λ as a parameter for the flows on $\mathbb{R}^n$, and for each l we define a flow ψ_λ on $\mathbb{R}^n$ by

$$(\psi_\lambda(t,x), \lambda) = \varphi^0(t,x,\lambda). \tag{A.2}$$

If we fix a range of values of λ, i.e., $\lambda \in \Lambda = [\lambda_0, \lambda_1]$, one can define a *parameterized flow*

$$\psi^\Lambda : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n \times \Lambda$$

by $\psi^\Lambda(t,x,\lambda) := (\psi_\lambda(t,x), \lambda)$.

Consider for the moment an arbitrary flow γ defined on a locally compact metric space X, a compact set $N \subset X$ is an *isolating neighborhood* if

$$\mathrm{Inv}(N,\gamma) := \{x \in X \mid \gamma(\mathbb{R}, x) \subset N\} \subset \mathrm{Int}N.$$

If $S = \mathrm{Inv}(N,\gamma)$ for some isolating neighborhood N, then S is referred to as an *isolated invariant set*. The Conley index is an index of isolating neighborhoods with the property that if $\mathrm{Inv}(N,\gamma) = \mathrm{Inv}(N',\gamma)$ then the Conley index of N equals the Conley index of N'. In this way one may, also, view the Conley index as an index of isolated invariant sets. We shall make use of the cohomological Conley index which is denoted by $CH^*(S)$ and is an Alexander-Spanier cohomology group.

As was mentioned earlier, given an isolating neighborhood its Conley index can be used to describe the dynamics of the associated isolated invariant set. In our case we will present theorems which can be used to prove the existence of heteroclinic orbits.

The first step is to find the appropriate isolating neighborhoods. This is done by choosing compact neighborhoods of the connecting orbits and segments of the branches of equilibria. Observe, however, that this cannot produce an isolating neighborhood under the singular flow φ^0. On the other hand, our interest is in the dynamics for $\varepsilon > 0$. Therefore, it is only important that the constructed neighborhood isolate under φ^ε when $\varepsilon > 0$.

The second step is to compute the Conley index of the isolating neighborhood for $\varepsilon > 0$. For that purpose, we will define building blocks for constructing an isolating neighborhood. The segments around the branches of equilibria are the simplest to define. Let ψ_λ be as in (A.2).

Definition A.1 $\mathcal{T} \subset \mathbb{R}^n \times \mathbb{R}$ is a *tube* if:

(1) There exists an interval $[a,b]$ such that $\mathcal{T} \subset \mathbb{R}^n \times [a,b]$ and $\mathcal{T}$ is an isolating neighborhood for

$$\begin{aligned} \psi^{\mathcal{T}} : \mathbb{R} \times \mathbb{R}^n \times [a,b] &\to \mathbb{R}^n \times [a,b], \\ (t,x,\lambda) &\mapsto (\psi_\lambda(t,x), \lambda). \end{aligned}$$

(2) There exists $\delta(\mathcal{T}) = \pm 1$ such that for all $(x,\lambda) \in \mathcal{T}$ we have $\delta(\mathcal{T})g(x,\lambda) > 0$.

We now turn to the neighborhoods of the connecting orbits and the non-trivial problem of how to relate the index information between the various tubes. The Conley index theory provides a variety of techniques for proving the existence of heteroclinic connections. We shall use the following. Recall that a *Morse decomposition*

$$\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$$

of an isolated invariant set S is a finite collection of disjoint compact invariant subsets $M(p)$, called *Morse sets*, indexed by a partially ordered set $(\mathcal{P}, >)$, with the property that; if $x \in S \setminus \bigcup_{p\in P} M(p)$, then there exist $q > p$ such that $\alpha(x)$, the alpha limit set of x, is contained in $M(q)$ and $\omega(x)$, the omega limit set of x, is contained in $M(p)$.

In the context of a parametrized flow $\psi^\Lambda : \mathbb{R} \times X \times \Lambda \to X \times \Lambda$, a Morse decomposition is said to *continue over* Λ if there is an isolated invariant set $S = \mathrm{Inv}(N, \psi^\Lambda)$ with a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$. Observe that if one defines

$$S_\lambda := S \cap (\mathbb{R}^n \times \{\lambda\}),$$

then S_λ is an isolated invariant set for ψ_λ. Similarly, $\{M_\lambda(p) \mid p \in (\mathcal{P}, >)\}$ is a Morse decomposition for S_λ. Since Morse sets are isolated invariant sets, $CH^*(M_\lambda(p))$ is defined. Furthermore, the index of each Morse set remains constant over Λ. Let $\lambda_0, \lambda_1 \in \Lambda$ and assume that

$$S_{\lambda_i} = \bigcup_{p\in\mathcal{P}} M_{\lambda_i}(p), \quad i = 0, 1.$$

Then, there exists a lower triangular (with respect to the order $>$) degree 0 isomorphism

$$T^{l_1,l_0} : \bigoplus_{p\in\mathcal{P}} CH^*(M_{l_1}(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_{l_0}(p))$$

called a *topological transition matrix* (see [8, 9]). Roughly, if the (p, q)-off diagonal entry of T^{l_1,l_0} is non-zero, then for some parameter value $l \in (l_0, l_1)$ there exists a connecting orbit between $M_l(p)$ and $M_l(q)$. As will become clear later, these off diagonal entries play a crucial role in the desired computation of the Conley index.

In order to insure the existence of topological transition matrices in the abstract setting of the fast-slow systems, we introduce the following neighborhoods of the connecting orbits.

Definition A.2 A set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}$ is a *box* if:

(1) There exists an interval $[c, d]$ such that $\mathcal{B} \subset \mathbb{R}^n \times [c, d]$ and $\mathcal{B}$ is an isolating neighborhood for the parameterized flow $\psi^{\mathcal{B}}$ defined by

$$\begin{aligned} \psi^{\mathcal{B}} : \mathbb{R} \times \mathbb{R}^n \times [c, d] &\to \mathbb{R}^n \times [c, d], \\ (t, x, \lambda) &\mapsto (\psi_\lambda(t, x), \lambda). \end{aligned}$$

(2) Let $S(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}, \psi^{\mathcal{B}})$. There exists a Morse decomposition

$$\mathcal{M}(S(\mathcal{B})) := \{M(p, \mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\},$$

with the usual ordering on the integers as the admissible ordering. Let $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathbb{R} \times \{\lambda\})$, $S_\lambda(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}_\lambda, \psi_\lambda)$ and let $\{M_\lambda(p, \mathcal{B}) \mid p = 1, \ldots, P_\mathcal{B}\}$ be the corresponding Morse decomposition of $S_\lambda(\mathcal{B})$. Then

$$S_c(\mathcal{B}) := \bigcup_{p=1}^{P_\mathcal{B}} M_c(p, \mathcal{B}) \quad \text{and} \quad S_d(\mathcal{B}) := \bigcup_{p=1}^{P_\mathcal{B}} M_d(p, \mathcal{B}).$$

(3) There are isolating neighborhoods $V(p, \mathcal{B})$ for $M(p, \mathcal{B})$ such that

$$V(p, \mathcal{B}) \subset \mathcal{B} \quad \text{and} \quad V(p, \mathcal{B}) \cap V(q, \mathcal{B}) = \emptyset$$

for $p \neq q$ and for every $\lambda \in [c, d]$

$$V_\lambda(p, \mathcal{B}) \subset \mathrm{Int}(\mathcal{B}_\lambda).$$

Furthermore, there are $\delta(p, \mathcal{B}) = \pm 1$, $p = 1, \ldots, P_\mathcal{B}$, such that

$$\delta(p, \mathcal{B}) g(x, \lambda) > 0 \qquad \text{for all } (x, \lambda) \in V(p, \mathcal{B}).$$

Notice that Definition A.2(2) implies that there are no connecting orbits between the Morse sets at the parameter values c and d, and by the construction, the sets $S_c(\mathcal{B})$ and $S_d(\mathcal{B})$ are related by continuation.

If one is attempting to prove the existence of heteroclinic orbits, an additional type of neighborhood which surrounds the critical points for the perturbed system is necessary.

Definition A.3 A set $\mathcal{C}(R)$ ($\mathcal{C}(A)$) is a *repelling (attracting) cap* if:

(1) There exists an interval $[e, f]$ such that $\mathcal{C} \subset \mathbb{R}^n \times [e, f]$ and $\mathcal{C}$ is an isolating neighborhood for

$$\begin{aligned} \psi^{\mathcal{C}} : \mathbb{R} \times \mathbb{R}^n \times [e, f] &\to \mathbb{R}^n \times [e, f] \\ (t, x, \lambda) &\mapsto (\psi_\lambda(t, x), \lambda) \end{aligned}$$

(2)

$$\begin{aligned} x \in \mathcal{C}_e(R) \quad &\Rightarrow \quad g(x, e) < 0 \\ x \in \mathcal{C}_f(R) \quad &\Rightarrow \quad g(x, f) > 0 \\ x \in \mathcal{C}_e(A) \quad &\Rightarrow \quad g(x, e) > 0 \\ x \in \mathcal{C}_f(A) \quad &\Rightarrow \quad g(x, f) < 0, \end{aligned}$$

where $\mathcal{C}_\lambda(R) := \mathcal{C}(R) \cap \{\lambda\}$ and $\mathcal{C}_\lambda(A) := \mathcal{C}(A) \cap \{\lambda\}$.

Finally, in order to construct a global isolating neighborhood, these boxes, tubes, and caps must be related in a consistent manner. The primary requirement is that the tubes and boxes overlap at the appropriate Morse sets. To simplify the notation we let $P_i = P_{\mathcal{B}_i}$ and $M(p, i) := M(p, \mathcal{B}(i))$.

Definition A.4 A *tubes, boxes and caps collection* (TBC collection) is a collection of tubes $\{\mathcal{T}(i) \,|\, i = 1, \ldots, I+1\}$, boxes $\{\mathcal{B}(i) \,|\, i = 1, \ldots, I\}$, and caps $\mathcal{C}(R)$ and $\mathcal{C}(A)$ such that:

(1) for $i = 1, \ldots, I$,

(a) $\mathcal{T}(i) \cap (\mathbb{R} \times [c_i, d_i]) \subset V(1, \mathcal{B}(i))$ and $\mathcal{T}(i) \cap \mathcal{B}(i)$ isolates $M(1, i)$.

(b) $\mathcal{T}(i+1) \cap (\mathbb{R} \times [c_i, d_i]) \subset V(P_i, \mathcal{B}(i))$ and $\mathcal{T}(i+1) \cap \mathcal{B}(i)$ isolates $M(P_i, i)$.

(2) for $i = 1, \ldots, I$, either

$$\delta(\mathcal{T}(i+1)) > 0 \text{ and } \delta(P_i, \mathcal{B}(i)) > 0 \text{ in which case } b_{i+1} = d_i$$

or

$$\delta(\mathcal{T}(i+1)) < 0 \text{ and } \delta(P_i, \mathcal{B}(i)) < 0 \text{ in which case } a_{i+1} = c_i$$

where a, b, c, and d are as in Definitions A.1 and A.2.

(3) for $i = 1, \ldots, I$, either

$$\delta(\mathcal{T}(i)) > 0 \text{ and } \delta(1, \mathcal{B}(i)) > 0 \text{ in which case } a_i = c_i$$

or

$$\delta(\mathcal{T}(i)) < 0 \text{ and } \delta(1, \mathcal{B}(i)) < 0 \text{ in which case } b_i = d_i$$

where a, b, c, and d are as in Definitions A.1 and A.2.

(4) If $i \neq j$, then $\mathcal{B}(i) \cap \mathcal{B}(j) = \emptyset$.

(5) $\mathcal{C}(R) \cap \mathcal{T}(I+1) \neq \emptyset$ and $\mathcal{C}(A) \cap \mathcal{T}(1) \neq \emptyset$. Furthermore,

$$\begin{aligned} \mathcal{C}(R) \cap \mathcal{T}(I+1) \cap (\mathbb{R}^n \times \{\lambda\}) \neq \emptyset \quad &\Rightarrow \quad \mathcal{C}_\lambda(R) = \mathcal{T}_\lambda(I+1), \\ \mathcal{C}(A) \cap \mathcal{T}(1) \cap (\mathbb{R}^n \times \{\lambda\}) \neq \emptyset \quad &\Rightarrow \quad \mathcal{C}_\lambda(A) = \mathcal{T}_\lambda(1). \end{aligned}$$

Given a TBC collection, let

$$T^i : \bigoplus_{p=1}^{\mathcal{P}_i} CH^*(M_{d_i}(p,i)) \to \bigoplus_{p=1}^{\mathcal{P}_i} CH^*(M_{c_i}(p,i))$$

denote the transition matrix associated with the box $\mathcal{B}(i)$ and let

$$T^i(P_i, 1) : CH^*(M_{d_i}(1,i)) \to CH^*(M_{c_i}(P_i,i)) \tag{A.3}$$

denote its corresponding entry. Again, having fixed the TBC collection, we define a map

$$\Theta := \Theta(I) = T^I(P_I, 1) \circ T^{I-1}(P_{I-1}, 1) \circ \ldots \circ T^2(P_2, 1) \circ T^1(P_1, 1). \tag{A.4}$$

This definition makes sense since the continuation of the Conley index allows for a natural identification between these spaces.

The following result can be used to find heteroclinic orbits. We begin with a concept concerning the dynamics within the isolating neighborhood.

The simplest non-trivial Morse decomposition of an isolated invariant set S consists of two Morse sets $M(1)$ and $M(0)$ with an admissible ordering $1 > 0$. In this case, $M(0)$ is called an *attractor* in S and $M(1)$ a *repeller*. Together, the pair $(M(0), M(1))$ is referred to as an *attractor repeller pair decomposition* of S.

Theorem A.5 *Let* $\{\mathcal{T}(i) \mid i = 1, \dots, I+1\}$, $\{\mathcal{B}(i) \mid i = 1, \dots, I\}$ *and* $\mathcal{C}(R), \mathcal{C}(A)$ *be a TBC collection. Let*

$$\mathcal{N} := \bigcup_{i=1}^{I} \mathcal{B}(i) \cup \bigcup_{i=1}^{I+1} \mathcal{T}(i) \cup \mathcal{C}(R) \cup \mathcal{C}(A).$$

Then, for $\varepsilon > 0$ *sufficiently small,*

(1) $\mathcal{N}$ *is an isolating neighborhood for* φ^{ε}*;*

(2) $(\mathrm{Inv}(\mathcal{C}(R), \varphi^{\varepsilon}), \mathrm{Inv}(\mathcal{C}(A), \varphi^{\varepsilon}))$ *is an attractor-repeller pair for* $\mathrm{Inv}(\mathcal{N}, \varphi^{\varepsilon})$*;*

(3) If $\Theta \neq 0$*, then*

$$CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^{\varepsilon})) \not\cong CH^*(\mathrm{Inv}(\mathcal{C}(A), \varphi^{\varepsilon})) \oplus CH^*(\mathrm{Inv}(\mathcal{C}(R), \varphi^{\varepsilon})).$$

In particular, for all sufficiently small $\varepsilon > 0$*, there is a connecting orbit from* $\mathrm{Inv}(\mathcal{C}(R), \varphi^{\varepsilon})$ *to* $\mathrm{Inv}(\mathcal{C}(A), \varphi^{\varepsilon})$ *in* $\mathcal{N}$ *under the flow* φ^{ε}*.*

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