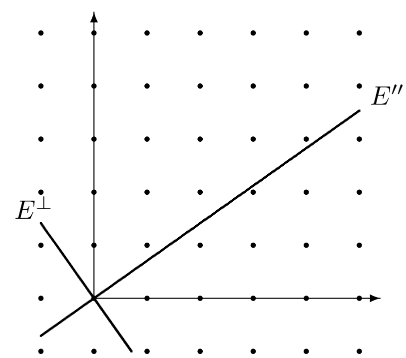
9 (9 February 2009)

Cut + Project Tilings

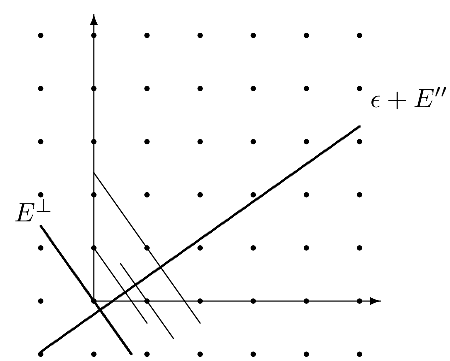
Canonical cut + project tiling.

$$\underset{=E^\perp}{\mathbb{R}^n} \overset{\text{proj.}}{\longleftarrow} \underset{\supset \mathbb{Z}^N \text{ lattice}}{\mathbb{R}^N} \overset{\text{proj.}}{\longrightarrow} \mathbb{R}^d =: E'' \quad (d < N)$$

$\tilde{\mathcal{T}}$ is a $\mathbb{Z}^N$-periodic tiling of $\mathbb{R}^N$ such that there are finitely many tiles up to translation, each of which is of the form $\mathsf{k} = \underset{\text{poly'n in } E^\perp}{\mathsf{k}^\perp} \times \underset{\text{poly'n in } E''}{\mathsf{k}''}$. ("$\mathsf{k}$" for the German "klotz")



Cut and Project: $N = 2\,,\ d = 1$



Cut and Project with $(\epsilon + E'') \cap \mathbb{Z}^2 = \emptyset$

Choose $\epsilon \in \mathbb{R}^N$: look at intersection between

$$(\epsilon + E'') \cap \tilde{\mathcal{T}}$$

Assumption $(\epsilon + E'') \cap \mathbb{Z}^2 = \emptyset$

if $\epsilon + E''$ cuts through $\mathbb{Z}^2$ then $\epsilon + E''$ cuts through the boundary of two neighboring tiles; want to avoid this so that's why we put the ϵ in.

Try to understand the inverse limit space.

$$f \in C_{0\text{-}\mathcal{P}\text{-e}}(\overset{E''}{\mathbb{R}}) \text{ defines a function } \tilde{f} \text{ on a dense set of } \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$$

$$C_{0\text{-}\mathcal{P}\text{-e}}(\mathbb{R}) \overset{i}{\subset} C(\underset{\text{0-cut}}{\mathbb{T}^2_{\text{0-c}}})$$

$$\underbrace{\text{Spec}\left(C(\mathbb{T}^2_{\text{0-c}})\right)}_{\cong \mathbb{T}^2_{\text{0-c}}} \overset{i^*|_{\text{Spec}}}{\longrightarrow} \text{Spec}\left(C_{0\text{-}\mathcal{P}\text{-e}}(\mathbb{R})\right), \quad x \in \mathbb{T}^2_{\text{0-c}} \mapsto \text{ev}_x$$

$$i^*\text{ev}_x = \text{ev}_x \circ i$$

$$i^*\text{ev}_x = i^*\text{ev}_y \text{ if } x \underset{\text{blue line}}{\sim} y \text{ or } x \underset{\text{green line}}{\sim} y$$

Delone set of $\mathcal{P}$ given by the vertices is

$$(\epsilon + E'') \cap (\mathbb{Z}^2 + \bigcup_{\mathsf{k} \text{ tile in } \tilde{\mathcal{T}}} \mathsf{k}^\perp)$$

Canonical choice: $\pi^\perp$(unit cube)

$$\pi^\perp : \mathbb{R}^N \to E^\perp \text{projection along } E''$$

The thing I think should be canonical

monical: $\mathsf{k}^\perp$ polyhedra s.th. $\partial \mathsf{k}^\perp = \bigcup_{\text{faces}} f_i$

Look at the stabilizer

$$\text{Stab}_{\pi^\perp(\mathbb{Z}^N)}(\text{affine space spanned by all } f_i > 0)$$

The rank of the stabilizer determines the complexity of the tiling.

Back to the construction of the algebra.

$$\mathcal{A}_\mathcal{P} = \underset{(\alpha_\xi(f))(x) = f(x-\xi)}{C_\mathcal{P}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n} \cong \underset{\phi \text{ induced by } \omega \in \Omega_\mathcal{P} \mapsto \omega - x}{C(\Omega_\mathcal{P}) \rtimes_\phi \mathbb{R}^n}$$

Fix isomorphism $C(\Omega_\mathcal{P}) \overset{\sigma}{\to} C_\mathcal{P}(\mathbb{R}^n)$, a $*$-isomorphism of C*-algebras: $\sigma(\tilde{f})(x) := \tilde{f}(\mathcal{P} - x)$

We can see why $C_\mathcal{P}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n$ is a good algebra.

Example.

$$\mathcal{P} = \{x\}\,,\ C_{\{x\}}(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)^{+^{(\text{adjoin } 1)}} = C_0(\mathbb{R}^n) + \underset{\text{multiples of } 1}{\mathbb{C}}$$

$$\mathcal{A}_{\{x\}} = \underset{\text{quant'n of a classical particle}}{C_0(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n} + \mathbb{C} \rtimes \mathbb{R}^n$$

Recall. We took the representation induced by $\mathrm{ev}_0 : C_{\{x\}}(\mathbb{R}^n) \to \mathbb{C}$. This gave the Schrödinger representation for the first part of $\mathcal{A}_{\{x\}}$.

$$\underset{\text{character rep'n}}{C_{\mathcal{P}}(\mathbb{R}^n)} \rtimes \underset{\text{induced rep'n}}{\mathbb{R}^n}$$

The representations on $\mathcal{A}_{\mathcal{P}}$ we look at are induced representations of the character irreducible representations on $C_{\mathcal{P}}$.

So $\forall \omega \in \Omega_{\mathcal{P}} = \mathrm{Spec}\,(C_{\mathcal{P}}(\mathbb{R}^n))$ we have an irreducible representation $\mathrm{ev}_\omega \circ \sigma^{-1}$ and therefore an irreducible representation π_ω of $\mathcal{A}_{\mathcal{P}}$.

Let $F \in \mathcal{A}_{\mathcal{P}}$ of the form $F : \mathbb{R}^n \to C_{\mathcal{P}}(\mathbb{R}^n)$

$$\begin{aligned}
\tilde{F}(\xi) &= \sigma^{-1} \circ F(\xi) \\
\tilde{F} &: \mathbb{R}^n \to C(\Omega_{\mathcal{P}}) \\
\pi_\omega &: \mathcal{A}_{\mathcal{P}} \to B(L^2(\mathbb{R}^n)) \\
(\pi_\omega(\tilde{F})\psi)(x) &= \int \mathrm{d}y \ \underbrace{\tilde{F}(x-y)(\omega+x)}_{\text{integral kernel of } \pi_\omega(\tilde{F})} \ \psi(y)
\end{aligned}$$

First important consequence of this:

Consider $\omega = \mathcal{P}$ and

$$H = \underset{\text{in the Sobolev rep'n}}{\frac{\hat{p}^2}{2m} + V(\hat{q})} \quad , V \in C_{\mathcal{P}}(\mathbb{R}^n, \mathbb{R})$$

Then $\forall g \in C_0(\mathbb{R})$,

$$\underset{\text{spec calc}}{g(H)} \in \pi_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}})$$

Use Laplace transform.