

## 10 (11 February 2009)

### The construction of the algebra $\mathcal{A}_{\mathcal{P}}$ (cont.)

We have this algebra

$$\begin{aligned}\mathcal{A}_{\mathcal{P}} &= C_{\mathcal{P}}(\mathbb{R}^n) \rtimes_{\alpha} \mathbb{R}^n \cong C(\Omega_{\mathcal{P}}) \rtimes_{\alpha} \mathbb{R}^n \\ \pi_{\mathcal{P}} : \mathcal{A}_{\mathcal{P}} &\rightarrow B(L^2(\mathbb{R}^n)) \\ f : \mathbb{R}^n &\rightarrow C_{\mathcal{P}}(\mathbb{R}^n) \\ (\pi_{\mathcal{P}}(f)\psi)(x) &= \int f(x-y)\psi(y)dy\end{aligned}$$

**Theorem.** *If  $H = \frac{\hat{p}^2}{2m} + V(\hat{q})$ ,  $V \in C_{\mathcal{P}}(\mathbb{R}^n)$ , then  $\forall F \in C_0(\mathbb{R})$ ,  $F(H) \in \pi_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}})$  ( $\Leftrightarrow \exists A \in \mathcal{A}_{\mathcal{P}}$  such that  $A$  is represented by  $F(H)$ ).*

*Proof.*  $\hat{p} = \frac{\hbar}{i}\Delta$ ,  $\hat{q}$  left multiplication by  $x \mapsto x$ . Taking  $\hbar = 1$ ,  $m = 1$ ,  $H = -\Delta + V$ , then  $H$  is bounded below (but not above), and by a shift of  $V$ ,  $H \geq 0$ .

Suppose  $F$  has a Laplace transform;  $F$  is regular enough that

$$\begin{aligned}\tilde{F}(t) &= 2\pi \int_{-\infty}^{\infty} e^{tE} F(E) dE \\ \left( F(E) = \int_0^{\infty} e^{-tE} \tilde{F}(t) dt \text{ is the Laplace transform of } \tilde{F} : \mathbb{R} \rightarrow \mathbb{R} \right)\end{aligned}$$

Then  $F(H)$  is defined by ‘‘Laplace’’ functional calculus

$$F(H) = \int_0^{\infty} e^{-tH} \tilde{F}(t) dt; \text{ bounded because } H \text{ is bounded below.}$$

To show that  $F$  belongs to a norm-closed subalgebra it suffices to show that  $e^{-tH}$  belongs; that is, for these functions  $F$ , it suffices to show that  $e^{-tH} \in \pi_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}})$ .

Consider first the case where

$$e^{-t(-\Delta)} \text{ (the heat kernel)}^1$$

**Lemma.**

$$\begin{aligned}(e^{t\Delta}\psi)(x) &= \int \underbrace{(e^{t\Delta})_{xy}}_{= \frac{c_1}{t^{n/2}} e^{-\frac{(x-y)^2}{4t}}} \psi(y) dy\end{aligned}$$

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<sup>1</sup>With  $\hbar = 1$ ,  $\hat{p}^2 = -\Delta$

$$\begin{aligned} \pi_{\mathcal{P}}(f) &= e^{t\Delta} & f : \mathbb{R}^n &\rightarrow C_{\mathcal{P}}(\mathbb{R}^n) \\ f(x-y)(x) & & f(\xi)(x) &= \frac{c_1}{t^{n/2}} e^{-\frac{\xi^2}{4t}} \end{aligned}$$

If we have a constant potential (zero potential)

$$e^{-tH} \in \underbrace{\mathbb{C} \times \mathbb{R}^n}_{\substack{\cong C_0(\mathbb{R}^n) \\ = \text{functions of } \hat{p}}} \subset \mathcal{A}_{\{x\}}$$

So [Dyson-Phillips expansion]

$$e^{-tH} = e^{-t\Delta - tV} = e^{t\Delta} + \sum_{\nu=0}^{\infty} \int ds_0 \cdots \int_{\substack{s_i \geq 0 \\ \sum_0^{\nu} s_i = t}} ds_{\nu} e^{s_1 \Delta} V e^{s_2 \Delta} V \cdots V e^{s_{\nu} \Delta}$$

If  $V$  is bounded, the series converges in norm.  $-\Delta \geq 0 \Rightarrow \|e^{s\Delta}\| \leq 1$

$$\left\| \int ds_0 \cdots \int_{\substack{s_i \geq 0 \\ \sum_0^{\nu} s_i = t}} ds_{\nu} e^{s_1 \Delta} \cdots \right\| \leq \int_{\substack{s_i \geq 0 \\ \sum_0^{\nu} s_i = t}} ds_1 \cdots ds_{\nu} \|V\|^{\nu} \leq \frac{t^{\nu}}{\nu!} \|V\|^{\nu}$$

volume of a  $\nu$ -simplex

So the series converges absolutely like  $\sum \frac{t^{\nu}}{\nu!} \|V\|^{\nu} \leq e^{t\|V\|}$

Now we are almost done. Need to show  $e^{s_i \Delta}$  is in the algebra, so that  $e^{s_1 \Delta} V e^{s_2 \Delta} V \cdots e^{s_{\nu} \Delta}$  is in the algebra.

It suffices to show that  $V e^{t\Delta} \in \pi_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}})$ :

$$\begin{aligned} (V e^{t\Delta})\psi(x) &= \int \underbrace{V(x) \frac{c_1}{t^{n/2}} e^{-\frac{|x-y|^2}{4t}}}_{\text{integral kernel } f(x-y)(x)} \psi(y) dy \\ & & f : \mathbb{R}^n &\rightarrow C_{\mathcal{P}}(\mathbb{R}^n) \\ & & f(\xi)(x) &= V(x) \frac{c_1}{t^{n/2}} e^{-\frac{\xi^2}{4t}} \\ & \text{then, } \pi_{\mathcal{P}}(f) &= V e^{t\Delta} \end{aligned}$$

QED □

**Q:** This is a good representation (Schrödinger); what about the others? (Other representations from other characters.)

**Covariant families of operators**

Comes from the physics of condensed matter. Random potentials are described by potentials which are random variables over some probability space  $(\Omega, \mathbb{P})$ . So

$$\Omega \ni \omega \mapsto V_\omega : \mathbb{R}^n \rightarrow \mathbb{R}$$

Ideally, solve everything for  $H_\omega = H_0 + V_\omega$  ( $H_0 = -\Delta$  for example).

$$\rightsquigarrow \int_{\Omega} \text{Tr}(\rho_\omega A_\omega) d\mathbb{P}$$

The density matrix might depend on  $\omega$ .

Homogeneous media = microscopically translation invariant.

Idea:  $(\Omega, \mathbb{P})$  should carry an action of  $\mathbb{R}^n$ :  $\omega \mapsto \omega - \xi$  such that

$$\forall \omega \in \Omega \ x \in \mathbb{R}^n : V_{\omega-x} = U_x V_\omega U_{-x}, \ ((U_x \psi)(y) = \psi(y-x))$$

Covariant system. Bellisard, Johnson and Moser: for aperiodic media take  $\Omega = \text{hull}$ .  $\mathbb{P}$  should be translation invariant. Pure phases = uniquely ergodic  $\mathbb{P}$  with respect to itself.