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Transport coefficients, cont.

<u>Result</u>: 1^{st}-order response to a perturbation $\mathcal{L}_{\text{per}} = \mathcal{L} + \vec{\lambda} \cdot \vec{\delta}$, where $\vec{\delta} = (\delta_1, \ldots, \delta_n)$ are commuting derivations and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ are complex constants.

Then

$$\langle \delta_\nu H \rangle_{\text{per}} = \sum_\mu \sigma^\delta_{\nu,\mu} \lambda_\mu + \mathcal{O}(\lambda^2) \quad (0^{\text{th}}\text{-order response} = 0)$$

defines the first order response (transport) coefficients

$$\sigma^\delta_{\nu\mu} = \text{C-}\lim_{t\to\infty} \tau \left(\rho \int_{\substack{s_0, s_1 \geq 0 \\ s_0 + s_1 = t}} \underbrace{\mathrm{e}^{-is_0\mathcal{L}}}_{\text{(put this onto } \rho\text{, but } \rho \text{ is invariant so cross this term off)}} \delta_\mu \mathrm{e}^{-is_1\mathcal{L}} \delta_\nu H \, \mathrm{d}s_0 \mathrm{d}s_1 \right)$$

$$= \text{C-}\lim_{t\to\infty} \tau \left(\int_{\substack{s_0, s_1 \geq 0 \\ s_0 + s_1 = t}} \underbrace{\left(\mathrm{e}^{-is_0\mathcal{L}}(\rho)\right)}_{\rho} \delta_\mu \mathrm{e}^{-is_1\mathcal{L}} \delta_\nu H \, \mathrm{d}s_0 \mathrm{d}s_1 \right)$$

$$= \text{C-}\lim_{t\to\infty} \tau \left(\delta_\mu \int_{\substack{s_0, s_1 \geq 0 \\ s_0 + s_1 = t}} \mathrm{e}^{-is_1\mathcal{L}} \delta_\nu H \, \mathrm{d}s_0 \mathrm{d}s_1 \right)$$

$$= 0 - \text{C-}\lim_{t\to\infty} \tau \left((\delta_\mu(\rho)) \int_0^t \mathrm{e}^{-is_1\mathcal{L}} \delta_\nu H \, \mathrm{d}s_1 \right)$$

because τ is invariant under derivation ($\tau \circ \delta = 0$), then $\tau(\delta_\mu(\rho \int \text{etc. })) = 0$

$$= -\tau\left(\delta_\mu(\rho) \mathcal{D}(\mathcal{L}) \delta_\nu(H)\right) \text{ (which may be infinite)}$$

$\lim_{t\to\infty} \int_0^t \mathrm{e}^{-is\omega} \mathrm{d}s \overset{\text{dist'n}}{=} \text{PV} \frac{1}{i\omega} + \frac{\pi}{2} \delta(\omega) =: \mathcal{D}(\omega)$

Attention if $\delta_\nu(H)$ is not $\perp$ to $\ker \mathcal{L}$.

Assumption $T \searrow 0$ and $\underset{\text{Fermi energy}}{E_F} \in \text{Gap}(H)$

Mathematical assumption is $\rho = P_F =$ the spectral projection of H onto states $\leq E_F$. This avoids the singularity.

Suppose we have $\{\psi_i\}_i$ an eigenbasis of H (in the generalized sense).

So $H\psi_i = E_i\psi_i$ (that is, $H\,|\,\psi_i\rangle = E_i\,|\,\psi_i\rangle$).

$$\langle\psi_i\,|\,\underset{\substack{| \\ \mathcal{L}(\,\cdot\,)=[H,\,\cdot\,]}}{\mathcal{L}}(\delta_\nu H)\,|\,\psi_j\rangle = \langle\psi_i\,|\,(H\delta_\nu H - (\delta_\nu H)H)\,|\,\psi_j\rangle$$
$$= (E_i - E_j)\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle$$

So,

$$\begin{aligned}\langle\psi_i\,|\,\mathcal{L}^n(\delta_\nu H)\,|\,\psi_j\rangle &= \langle\psi_i\,|\,\mathcal{L}(\mathcal{L}^{n-1}(\delta_\nu H))\,|\,\psi_j\rangle \\ &= (E_i - E_j)\langle\psi_i\,|\,\mathcal{L}^{n-1}(\delta_\nu H)\,|\,\psi_j\rangle\end{aligned}$$

Hence

$$\langle\psi_i\,|\,\mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\,|\,\psi_j\rangle = \mathrm{e}^{-is_1(E_i-E_j)}\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle$$

So we need that $E_i \neq E_j$ (to avoid $\omega = 0$ in the distribution $\mathcal{D}(\omega)$). In that case

$$\langle\psi_i\,|\,\lim_{t\to\infty}\int_0^t \mathrm{e}^{-is_1\mathcal{L}}\mathrm{d}s_1\delta_\nu H\,|\,\psi_j\rangle = \frac{1}{i(E_i - E_j)}\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle$$

Since $\rho = P_F$ is a projection and δ_μ is a derivation (using the beautiful formula $(*)$ from the previous lecture)

$$\delta_\mu P_F = P_F(\delta_\mu P_F)P_F^\perp + P_F^\perp(\delta_\mu P_F)P_F$$

Now

$$\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle \neq 0 \Leftrightarrow ((E_i \le E_F) \wedge (E_j > E_F)) \vee ((E_j \le E_F) \wedge (E_i > E_F))$$

Since $E_F \in (\tilde{E}_0, \tilde{E}_1)$, a gap in $\sigma(H)$, we have

$$|E_i - E_j| \ge |\tilde{E}_1 - \tilde{E}_0|$$

As a consequence,

$$\begin{aligned}\sigma^\delta_{\nu\mu} &= -\tau\,(\delta_\mu(\underset{\substack{1=P_F+P_F^\perp \\ \vee}}{P_F})\mathcal{D}(\mathcal{L})\delta_\nu(H)) \\ &= -\tau\,(P_F\delta_\mu(P_F)(\mathcal{D}\mathcal{L})\delta_\nu(H)) - \tau\left(P_F^\perp\delta_\mu(P_F)\mathcal{D}(\mathcal{L})\delta_\nu(H)\right) \\ &= -\tau\,(P_F\delta_\mu(P_F)(\mathcal{D}(\mathcal{L})\delta_\nu(H))P_F) - \tau\left(P_F^\perp\delta_\mu(P_F)(\mathcal{D}(\mathcal{L})\delta_\nu(H))P_F^\perp\right) \\ &= -\tau\left(P_F\delta_\mu(P_F)P_F^\perp\frac{1}{i}\mathcal{L}^{-1}(\delta_\nu H)P_F\right) - \tau\left(P_F^\perp\delta_\mu(P_F)P_F\frac{1}{i}\mathcal{L}^{-1}(\delta_\nu H)P_F^\perp\right) \\ &\overset{\text{claim}}{=} -i\tau\,(P_F\delta_\mu(P_F)\delta_\nu(P_F) - \delta_\nu(P_F)\delta_\mu(P_F)) \\ &\overset{\text{or}}{=} -i\tau\left(P_F\delta_\mu(P_F)P_F^\perp\delta_\nu(P_F)P_F - \delta_\nu(P_F)\delta_\mu(P_F)\right)\end{aligned}$$

In other words the claim is

$$P_F^{\perp}\mathcal{L}^{-1}(\delta_\nu H)P_F = -P_F^{\perp}(\delta_\nu P_F)P_F \quad \text{and}$$
$$P_F\mathcal{L}^{-1}(\delta_\nu H)P_F^{\perp} = P_F(\delta_\nu P_F)P_F^{\perp}$$

Proof. Hit both sides of the first equation with $\mathcal{L}$ ($\mathcal{L}$ leaves P_F invariant).

left-hand side: $\mathcal{L}\left(P_F^{\perp}\mathcal{L}^{-1}(\delta_\nu H)P_F\right) = P_F^{\perp}(\delta_\nu H)P_F$

right-hand side: $\mathcal{L}\left(-P_F^{\perp}(\delta_\nu P_F)P_F\right) = -P_F^{\perp}[H,\delta_\nu P_F]P_F$

$$\underset{\mathcal{L}(\,\cdot\,)=[H,\,\cdot\,]}{|}$$

$$= -P_F^{\perp}\left(\delta_\nu\left([H,P_F]\right) - [\delta_\nu H, P_F]\right)P_F$$

$$\underset{[H,P_F]=0}{|}$$

$$= P_F^{\perp}\left((\delta_\nu H)P_F - P_F(\delta_\nu H)\right)P_F$$
$$= P_F^{\perp}(\delta_\nu H)P_F \quad \text{(QED first equation)}$$

The proof of the second equation is similar and picks up an extra minus sign. □

So this is the result

$$\sigma^{\delta}_{\nu\mu} = i\,\tau(P_F\,[(\delta_\nu P_F),(\delta_\mu P_F)])$$

Consequence: $\sigma^{\delta}_{\nu\mu}$ *is a topological invariant.*

Remark. $\sigma^{\delta}_{\nu\mu}$ is anti-symmetric.

Example. QHE in $\mathbb{R}^2$ (the quantum Hall effect).

$$\delta = (\delta_1.\delta_2) \qquad \delta_\nu = [\hat{q}_\nu,\,\cdot\,]$$
$$\lambda = (\lambda_1,\lambda_2) \qquad \lambda_\nu = e\,E_\nu$$

e is the electric charge

E_ν is the external electric field