Lecture 18 (4 March 2009)

Cyclic cohomology and higher traces.

To describe topologically quantized transport coefficients we will use $\underset{\text{algebra}}{\mathcal{A}}$, $\underset{\text{trace}}{\tau}$, $\underset{\text{comm. derivns on } \mathcal{A}}{\vec{\delta} = (\delta_1, \ldots, \delta_n)}$ such that $\tau \circ \delta_\nu = 0 \ \forall \nu$.

This is the realm of *higher traces* on Banach algebras.

Let $\mathcal{B}$ be an associative algebra over k (field). Let $C^n_\lambda(\mathcal{B}) = \{\text{cyclic } n{+}1\text{-forms}\}$. So $\eta \in C^n_\lambda(\mathcal{B})$ this is a map

$$\eta : \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{n+1} \to \mathsf{k}$$

which is linear in each argument and *cyclic*:

$$\eta(A_0, \ldots, A_n) = (-1)^n \eta(A_1, \ldots, A_n, A_0)\,, \quad A_i \in \mathcal{B}$$

$C^n_\lambda(\mathcal{B})$ is a VS over k.

Define a differential operator

$$\begin{aligned} \mathtt{b} : C^n_\lambda(\mathcal{B}) &\to C^{n+1}_\lambda(\mathcal{B})\,, \text{where} \\ \eta \in C^n_\lambda(\mathcal{B}) &\longmapsto \\ \mathtt{b}\eta(A_0, \ldots, A_{n+1}) &= \sum_{i=0}^{n} (-1)^i \eta(A_0, \ldots, A_i \cdot A_{i+1}, \ldots, A_{n+1}) \\ &\quad + (-1)^{n+1} \eta(A_{n+1} \cdot A_0, A_1, \ldots, A_n) \end{aligned}$$

Lemma (Exercise)**.** $\mathtt{b} \circ \mathtt{b} = 0$. *The differential complex* $(C^n_\lambda(\mathcal{B}), \mathtt{b})$ *is a subcomplex (because of cyclic) of the complex for the Hochschild cohomology* $H(\mathcal{B}, \mathcal{B}^*)$ *(of* $\mathcal{B}$ *with coefficients in* $\mathcal{B}^*$*).*

Definition. *Cyclic cohomology of* $\mathcal{B}$ is the cohomology of $(C^n_\lambda(\mathcal{B}), \mathtt{b})$, denoted $H^nC(\mathcal{B})$. $\eta \in C^n_\lambda(\mathcal{B})$ is a *cyclic cocycle* if $\eta \in \operatorname{Ker} \mathtt{b}$.

Ex 1: $\mathcal{B}$ a C*-algebra and τ a bounded trace on $\mathcal{B}$.

$$\tau \in C^0_\lambda(\mathcal{B}) \Rightarrow \mathtt{b}\tau(A_0, A_1) = \tau(A_0 A_1) - \tau(A_1 A_0) = 0$$

since traces are cyclic. So a bounded trace is a cyclic 0-cocycle.

Ex 2: $\mathcal{M}$ an n-dim smooth, compact manifold without boundary, $\mathcal{B} = C^\infty(\mathcal{M})$ and for $\tau \in C^n_\lambda(\mathcal{B})$ (d is the exterior derivative on $\mathcal{M}$):

$$\tau(f_0, \ldots, f_n) := \int_{\mathcal{M}} f_0 \, \mathrm{d} f_1 \, \mathrm{d} f_2 \cdots \mathrm{d} f_n \quad ((n+1)\text{-linear form})$$

$$f_0 \,\mathrm{d}\, f_1 \cdots \mathrm{d}\, f_n = (-1)^{n-1} \underbrace{(\mathrm{d}\, f_n) f_0}_{\mathrm{d}(f_n f_0) - f_n \,\mathrm{d}\, f_0} \;\mathrm{d}\, f_1 \cdots \mathrm{d}\, f_{n-1} \text{ (sign from exterior product)}$$

$$= (-1)^{n-1} \,\mathrm{d}(f_n f_0)\,\mathrm{d}\, f_1 \cdots \mathrm{d}\, f_{n-1} + (-1)^n f_n \,\mathrm{d}\, f_0 \cdots \mathrm{d}\, f_{n-1}$$

$$\Longrightarrow$$

$$\int_{\mathcal{M}} f_0 \,\mathrm{d}\, f_1 \cdots \mathrm{d}\, f_n = 0 + (-1)^n \int_{\mathcal{M}} f_n \,\mathrm{d}\, f_0 \cdots \mathrm{d}\, f_{n-1}$$

$\mathrm{d}(f_n f_0)\,\mathrm{d}\, f_1 \cdots \mathrm{d}\, f_{n-1}$ is exact and there is no boundary

So τ is cyclic.

Now

$$\begin{aligned}
\mathtt{b}\tau(f_0, \ldots, f_{n+1}) &= \sum_{i=0}^{n} (-1)^i \tau(f_0, \ldots, f_i f_{i+1}, \ldots, f_{n+1}) \\
&\quad + (-1)^{n+1} \tau(f_{n+1} f_0, f_1, \ldots, f_n) \\
&= \int_{\mathcal{M}} (f_0 f_1)\,\mathrm{d}\, f_2 \,\mathrm{d}\, f_3 \cdots \mathrm{d}\, f_{n+1} - \int_{\mathcal{M}} f_0 \,\mathrm{d}(f_1 f_2)\,\mathrm{d}\, f_3 \cdots \mathrm{d}\, f_{n+1} \\
&\quad + \int_{\mathcal{M}} f_0 \,\mathrm{d}\, f_1 \,\mathrm{d}(f_2 f_3) \cdots \mathrm{d}\, f_{n+1} - \cdots \\
&\quad + (-1)^n \int_{\mathcal{M}} f_0 \,\mathrm{d}\, f_1 \,\mathrm{d}\, f_2 \cdots \mathrm{d}(f_n f_{n+1}) + (-1)^{n+1} \int_{\mathcal{M}} (f_{n+1} f_0)\,\mathrm{d}\, f_1 \,\mathrm{d}\, f_2 \cdots \mathrm{d}\, f_n
\end{aligned}$$

and from this expansion it follows that $\mathtt{b}\tau(f_0, \ldots, f_{n+1}) = 0$ as follows: in each integral except the first and last expand $\mathrm{d}(f_i f_{i+1}) = f_i \,\mathrm{d}\, f_{i+1} + (\mathrm{d}\, f_i) f_{i+1}$; the resulting $2n + 2$ integrals cancel in pairs (exercise).

So, τ *is a cyclic* n*-cocycle*

Result: almost, $HC^n\left(C^\infty(\mathcal{M})\right) = H^n_{deRahm}(\mathcal{M})$.

Little catch: $C^\infty(\mathcal{M})$ is not a C*-algebra.

Reassuring: de Rahm cohomology ≅ Čech-cohomology and therefore is purely topological.

The pairing with K-theory

A cyclic cocycle over a C*-algebra $\mathcal{A}$ (over $\mathbb{C}$) defines a homomorphism

$$K_0(\mathcal{A}) \to \mathbb{C}$$

Suppose $\mathcal{A}$ is unital. Recall that $K_0(\mathcal{A})$ is made from homotopy classes of projections in $M_k(\mathcal{A})\,\forall k$. First extend a cyclic cocycle η of $\mathcal{A}$ to one, $\tilde{\eta}$, on $M_k(\mathcal{A})$. Let Tr be the standard trace on $M_k(\mathbb{C})$. Then, with $M_i \in M_k(\mathbb{C})$ and $A_i \in \mathcal{A}$, define

$$\tilde{\eta}(M_0 \otimes A_0, M_1 \otimes A_1, \ldots, M_n \otimes A_n) = \mathrm{Tr}(M_0 M_1 \cdots M_n)\eta(A_0, \ldots, A_n)$$

with linear extension to the general element of $M_k(\mathcal{A})$. Then $\tilde{\eta}$ is cyclic.

Proposition. *Let p and q be homotopic projections in $M_k(\mathcal{A})$, and let η be a cyclic n-cocycle. Then*

$$\tilde{\eta}(\underbrace{p, p, \ldots, p}_{n+1}) = \tilde{\eta}(\underbrace{q, q, \ldots, q}_{n+1})$$

Remark (Marcy). If n is odd then, since $\tilde{\eta}$ is cyclic, $\tilde{n}(p, \ldots, p) = -\tilde{n}(p, \ldots, p)$, and therefore $\tilde{n}(p, \ldots, p) = 0$.

Proof of the Proposition. Suppose $n = 2m$ and let $m = 1$ (other values of m are left as an exercise). Suppose there is a differentiable homotopy $p(t)$ between p and q, so that $p(0) = p$ and $p(1) = q$.

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\eta(p(t), \ldots, p(t)) = \eta(\dot{p}, p, \ldots, p) + \eta(p, \dot{p}, \ldots, p) + \cdots = (n+1)\eta(\dot{p}, p, \ldots, p)$$

From the 27 Feb. lecture,

$$\dot{p} = p\dot{p}p^{\perp} + p^{\perp}\dot{p}p$$

Now for $m = 1$,

$$\begin{aligned}\eta(p\dot{p}p^{\perp}, p, p) = {} & \underbrace{\mathtt{b}\eta(p\dot{p}, p^{\perp}, p, p)}_{=0 \text{ by cocycle hypoth.}} + \underbrace{\eta(p\dot{p}, p^{\perp}p, p)}_{p^{\perp}p=0} \\ & \underbrace{-\eta(p\dot{p}, p^{\perp}, p^2) + \eta(p^2\dot{p}, p^{\perp}, p)}_{\text{cancel because } p^2=p} \\ = {} & 0\end{aligned}$$

And a similar argument shows that $\eta(p^{\perp}\dot{p}p, p, p) = 0$, so $\eta(\dot{p}, p, p) = 0$, and therefore $\eta(p(t), \ldots, p(t))$ is constant. $\square$

This allows us to define η and $\tilde{\eta}$ on $K_0(\mathcal{A})$.