19 (6 March 2009)

Cyclic cohomology and higher traces, cont.

If η is a cyclic cocycle

$$\langle \eta \,|\, [p]_0 \rangle = c_n \eta \overbrace{(p, \ldots, p)}^{n+1}, \quad c_n = c_{2k} = \frac{1}{(2\pi i)^k} \frac{1}{k!}$$

is well-defined.

Aside (exercise): if $\eta = \mathtt{b}\tau$, then $\langle \mathtt{b}\tau \,|\, [p]_0 \rangle = 0$.

So one has a bilinear map $HC^n(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}$.

Ex.: $HC^n(\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

Smells like Bott periodicity.

Theorem. *If $\mathcal{A}$ is a C^*-algebra, then* $HC^n(\mathcal{A}) = \begin{cases} \textit{bounded traces on } \mathcal{A} & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$

Nothing interesting for C*-algebras.

So we need to include smoothness.

Let $\mathcal{A} = C(\mathcal{M})$; $\mathcal{M}$ an even dimensional, smooth, compact manifold without boundary.

$$\mathcal{A}^\infty = C^\infty(\mathcal{M}) \quad \text{in its Fréchet topology}$$

Require cyclic cocycles to be continuous w.r.t this topology.

Lemma. *For any projection $p \in \mathsf{M}_n(C(\mathcal{M}))$ there exists $\tilde{p} \in \mathsf{M}_n(C^\infty(\mathcal{M}))$ such that $p \sim \tilde{p}$.*

Then one could consider cyclic cocycles of $C^\infty(\mathcal{M})$ (in particular, C^∞-continuous) and extend the pairing with $K_0(C^\infty(\mathcal{M}))$ to a pairing with $K_0(C(\mathcal{M}))$

Purpose: to construct cyclic cocycles (continuous) for a dense subalgebra $\mathcal{A}^\infty$ of $\mathcal{A}$ such that the pairing extends from $K_0(\mathcal{A}^\infty)$ to $K_0(\mathcal{A})$.

Cycles. Look at $(\Omega, d, \int)$, a *cycle over a Banach algebra $\mathcal{B}$* of dimension n:

$$\Omega \text{ a graded algebra, } \Omega = \bigoplus_{i\in\mathbb{N}_0} \Omega^i,\ \Omega^i\Omega^j \subset \Omega^{i+j},\ \Omega^{n+p} = 0\ \forall\, p > 0\,;$$

$$d \text{ a differential of degree } +1\,, d^2 = 0 \;+\; \text{Leibniz rule;}$$

$$\int \text{ is a } \textit{graded} \text{ trace[9] on } \Omega^n\,, \text{ that is, } \int \text{ is linear and}$$

$$\int \omega_1\omega_2 = (-1)^{k(n-k)} \int \omega_2\omega_1\,,\ \omega_1 \in \Omega^k\,,\ \omega_2 \in \Omega^{n-k}\,;$$

and lastly, $\mathcal{B}$ is a subalgebra of Ω^0.

Ex.: $\mathcal{B} = C^\infty(\mathcal{M})$, $\Omega = \Omega(\mathcal{M})$ the algebra of exterior forms over $\mathcal{M}$, d the exterior derivative, and $\int$ the integral of $(\dim \mathcal{M})$-forms over $\mathcal{M}$.

Proposition (Connes)**.** *Any cycle over $\mathcal{B}$ of dimension n defines a cyclic n-cocycle η (the character of the cycle):*

$$\eta(A_0, \ldots, A_n) = \int A_0 dA_1 dA_2 \cdots dA_n$$

Conversely, any cyclic n-cocycle arises in this way.

Definition. An *n-trace* on a Banach algebra $\mathcal{B}$ is the character of a cycle of dimension n, $(\Omega', d, \int)$, over a dense subalgebra $\mathcal{B}'$ of $\mathcal{B}$, such that $\forall\, A_1, \ldots, A_n \in \mathcal{B}'$, $\exists\, C(A_1, \ldots, A_n)$ for which

$$\int (X_1 dA_1)(X_2 dA_2) \cdots (X_n dA_n) \le C\|X_1\|\|X_2\| \cdots \|X_n\|\,,\ \forall\, X_i \in \mathcal{B}'$$

This means that $\forall\, A_1, \ldots, A_n \in \mathcal{B}'$

$$\underbrace{\mathcal{B}' \times \cdots \times \mathcal{B}'}_{\text{n}} \to \mathbb{C} \quad n\text{-linear map}$$

$$(X_1, \ldots, X_n) \mapsto \int (X_1 dA_1)(X_2 dA_2) \cdots (X_n dA_n)$$

is bounded with norm $p(A_1, \ldots, A_n) :=$ the smallest $C(A_1, \ldots, A_n)$.

Theorem (Connes)**.** *Any n-trace on $\mathcal{B}$ extends to an algebra $\mathcal{B}''$, $\mathcal{B}' \subset \mathcal{B}'' \subset \mathcal{B}$, such that the inclusion $\mathcal{B}'' \xrightarrow{i} \mathcal{B}$ induces an isomorphism i_* in K-theory: $i_* : K_i(\mathcal{B}'') \underset{\cong}{\to} K_i(\mathcal{B})$.*

Consequence: An n-trace defines a functional on $K_i(\mathcal{B})$; first on $K_i(\mathcal{B}'')$ by continuous extension, and then by selecting for a dense-in-$\mathcal{B}$ representation of $\mathcal{B}''$.

[9] In addition, $\int$ is *closed:* $\int d\omega = 0$ for $\omega \in \Omega^{n-1}$. See the notes for 9 March.

Example.

$$\begin{array}{lll} \mathcal{A} = \text{ a C*-algebra,} & \tau = \text{ a trace,} & \delta = \text{ a derivation} \\ \updownarrow & \updownarrow^{\text{unbounded}} & \updownarrow^{\text{unbounded}} \\ \Omega = \mathcal{A} \otimes \bigwedge \mathbb{C}^2 & \int & d \end{array}$$