38 (1 May 2009)

Another Gap Labelling[32]

Rotation Numbers (Moser, 1-D).

Integrated Density of States (any dimension).

Now K_n-gap labelling (any dimension).

1-D differential operator $H = -\partial^2 + V$ (unit $\hbar^2/2m = 1$), where V is a differentiable function on $\mathbb{R}$.

Consider $E \in \mathbb{R}$,

$$(*) \qquad H\psi = E\psi\,,\ \psi : \mathbb{R} \to \mathbb{C} \quad (\psi \text{ non-zero})$$

Always solvable for $\psi \neq 0$ but the solution depends on E; this is where Moser started.

1) If E is an eigenvalue $\exists\, \psi \in L^2(\mathbb{R})$ solving $(*)$.

2) If $E \in \sigma(H)$ but not an eigenvalue:

   Simple if $\sigma(H) = \sigma_{\substack{\text{abs}\\ \text{cont}}}(H) \cup \sigma_{\text{pure point}}(H)$; then ψ is bounded $\rightsquigarrow$ linear combinations of these to construct wave packets. This is difficult.

   If $E \in \sigma_{\text{sc}}(H)$ "ψ is critical". This was the devil until the 80's; then became fashionable.

   Remark: If $V = 0$, then $\sigma = \mathbb{R} = \sigma_{\text{ac}}$ then use Fourier transforms to construct wave packets. If $V \neq 0$ then?

3) If $E \notin \sigma(H)$ then $\exists!$ (up to multiplicative constants) solutions ψ_+, ψ_- which satisfy $\lim_{x\to\pm\infty} \psi_\pm(x) = 0$ and $x \to \mp\infty$ leads to exponential increase of $\psi_\pm$ (see Fig. 20). This excludes any possible interpretation of

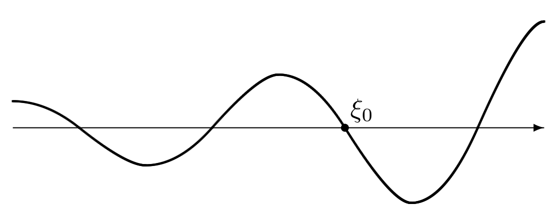

Figure 20: ψ_-

these particles. The only way is to restrict the system.

[32] Thanks to Carl Olimb for the notes.

Rotation Numbers: $E \notin \sigma(H)$; $\alpha(E) =$ "density of zeros" of $\psi_- = 2$ density of rotation of $\psi(x) + i\psi'(x)$. $\alpha(E)$ is constant on gaps.

What is the density in this context? We mean a sequence of intervals subdivided and take rotation numbers. So may depend on this sequence.

$$[a_n, b_n] \subset [a_{n+1}, b_{n+1}] \subset \cdots \to \mathbb{R}$$

The number of zeros in the interval is equal to the number of density states. Then

$$\alpha(E) = \text{IDS}(E) = \langle \underset{\ni \tau}{\text{0-Hull}} \,|\, [P(H)]_0 \rangle \,,\ K_0\text{-label.}$$

Now $H_\xi = -\partial^2 + V_\xi$, $V_\xi(x) = V(x+\xi)$. $H_\xi \underset{\text{unitary}}{\sim} H$, so $\sigma(H_\xi) = \sigma(H)$. If $E^{(\text{fixed})} \notin \sigma(H)$, $H_\xi \psi_{\xi -} = E\psi_{\xi -}$ with $\psi_{\xi -}(x) \overset{x \to -\infty}{\longrightarrow} 0$. Then $\psi_{\xi -}(x) = \psi_-(x+\xi)$. Consider Fig. 20 again:

$$\begin{aligned} &\xi_0 \text{ is a zero of } \psi_- \\ \Leftrightarrow\ &0 \text{ is a zero of } \psi_{\xi_0 -} \\ \Leftrightarrow\ &E \text{ is an eigenvalue of } H_\xi \,|_{\mathbb{R}^-} \\ &\quad \text{with Dirichlet boundary condition at } 0 \end{aligned}$$

Call E a *Dirichlet value of* H_{ξ_0}.

Notation:

$$\hat{H}_{\xi_0} := H_{\xi_0} \,|_{\mathbb{R}^-} \ \text{ with Dirichlet boundary condition.}$$

Write $D_\xi(0) = D_\xi = \{\text{Dirichlet values of } H_\xi \text{ in } \Delta\}$, where Δ is a gap in $\sigma(H_\xi)$ $= \sigma(H)$.

General remarks:

i) $\sigma_{\text{ac}}(\hat{H}_\xi) \subset \sigma_{\text{ac}}(H_\xi)$. So, $\sigma_{\text{ac}}(\hat{H}_\xi) \subset \sigma(H) \cup \bigcup_{\text{gaps } \Delta} D_\xi(\Delta)$

ii) If $\alpha(\Delta) \neq 0$, then $|D_\xi(\Delta)| \leq 1$. "No second eigenvalue in the same gap."

Let $\mu(\xi) =$ EV of $\hat{H}_\xi$ in Δ.

$$(**) \qquad \{\text{zeros of } \psi_-\} = \{\xi \,|\, |D_\xi(\Delta)| = 1\}$$

Now vary ξ; we get curves as in Fig. 21.

Lemma 4. $\mu'(\xi) < 0$

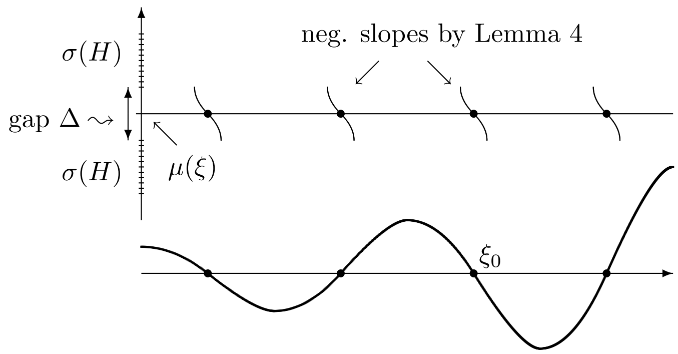

Figure 21: μ

Proof. $\mu(\xi) = \langle \hat{\psi}_\xi \,|\, \underbrace{\hat{H}_\xi}_{-\partial^2 + V_\xi} \,|\, \hat{\psi}_\xi \rangle$

$$\frac{\partial \mu(\xi)}{\partial \xi} = \underbrace{\langle \hat{\psi}'_\xi \,|\, \hat{H}_\xi \,|\, \hat{\psi}_\xi \rangle + \langle \hat{\psi}_\xi \,|\, \hat{H}_\xi \,|\, \hat{\psi}'_\xi \rangle}_{\mu(\xi)(\langle \hat{\psi}_\xi, \hat{\psi}'_\xi \rangle + \langle \hat{\psi}'_\xi, \hat{\psi}_\xi \rangle) = \frac{\partial}{\partial \xi} |\hat{\psi}_\xi|^2 = 0} + \underbrace{\langle \hat{\psi}_\xi \,|\, V'_\xi \,|\, \hat{\psi}_\xi \rangle}_{\uparrow}$$

expectation of mechanical force of this wave funtion

$$\begin{aligned} &= \int_{-\infty}^{0} \hat{\psi}_\xi V'_\xi \hat{\psi}_\xi \,\mathrm{d}\,x = - \int_{-\infty}^{0} \left(\hat{\psi}'_\xi V_\xi \hat{\psi}_\xi + \hat{\psi}_\xi + \hat{\psi}_\xi V_\xi \hat{\psi}'_\xi \right) \mathrm{d}\,x \\ &= \int_{-\infty}^{0} \left(\hat{\psi}'_\xi \partial^2 \hat{\psi}_\xi + \hat{\psi}_\xi \partial^2 \hat{\psi}'_\xi \right) \mathrm{d}\,x \\ &= - \int_{-\infty}^{0} \frac{\partial}{\partial x} (\hat{\psi}'_\xi \hat{\psi}'_\xi) \,\mathrm{d}\,x = -|\hat{\psi}'_\xi(0)|^2 < 0 \end{aligned}$$

□

Two important results: $-|\hat{\psi}'_\xi(0)|^2 < 0$ and the resultant force.

In our Fig. 21 the curves cannot cross since then we would have a degenerate eigenvalue. Now look at the "line" created by $\mu(\xi)$ on a circle.

$$\tilde{\mu} : \mathbb{R} \to \{ z \in \mathbb{C} \,|\, |z| = 1 \}, \text{ where } \tilde{\mu}(\xi) = \exp\left(2\pi i \, \frac{\mu(\xi) - E_0}{|\Delta|} \right), \; E_0 = \inf \Delta$$

where $\mu(\xi) = E_0$ is no eigenvalue at ξ.

<u>Final Result</u>. Moser rotation number is the same as another rotation number.

$$(**) \qquad \begin{aligned} \alpha(E) &= \text{ density of zeros of } \psi_- \\ &= \text{ density of intersections of } \mu \text{ with } E \\ &= \text{ density of } \tilde{\mu} = \beta(E) = \beta(\Delta) \end{aligned}$$

$$\beta(\Delta) = \lim_{[a_n,b_n]\nearrow\mathbb{R}} \frac{\Delta}{|b_n - a_n|}\frac{\Delta}{2\pi i}\int_{a_n}^{b_n} \mu^*(\xi)\mu(\xi)\,\mathrm{d}\,\xi$$
$$= \lim_{[a_n,b_n]\nearrow\mathbb{R}} \frac{\Delta}{|\Delta|}\int_{a_n}^{b_n} \mathrm{Tr}(-V_\xi' P_\Delta(\hat{H}_\xi))\,\mathrm{d}\,\xi$$

where $P_\Delta(\hat{H})$ is the spectral projection to states in Δ (actually only one); so $P_\Delta(\hat{H}) = |\,\psi_\xi\rangle\langle\hat{\psi}_\xi\,|$. This can be interpreted as a pairing of a 1-trace with a K_1-class.

$$U_\xi = \mathrm{e}^{2\pi i \frac{\hat{H}_\xi - E_0}{|\Delta|}} P_\Delta(\hat{H}_\xi) + P_\Delta(\hat{H}_\xi)^\perp \in \mathcal{K}(L^2(\mathbb{R}))^+$$
$$U_\Delta := (\xi \mapsto U_\xi) \in C_0(\mathbb{R}, \mathcal{K}(L^2(\mathbb{R}))^+) = \mathrm{S}\,\mathbb{C}\otimes\mathcal{K}^+ = C(\mathrm{S}^1)\otimes\mathcal{K}^+$$

so $[U_\Delta] \in K_1(C_0(\mathbb{R}))$.

Now 1-trace = character of $(\underbrace{\Omega(\mathbb{R})}_{\text{DeRahm forms}}, d, \int_{\mathbb{R}})$. Then $\langle \eta \,|\, [U_\Delta]\rangle = \beta(\Delta)$.

To finish, $\alpha = \beta$ is an index theorem. Why?

$$[U_\Delta]_1 = \exp[\underset{\substack{\uparrow\\ \text{energy values } \leq E}}{P_{\leq E}}(H)]_0 \text{ and additional work}$$