1 (21 January 2009)

1.1 Noncommutative Topology and Applications to Physics

Overview

Noncommutative topology (NCT) is the topology of NC spaces (NC C*-algebras or other types of NC algebras)

Topology = basically, algebraic topology

Main tools: K-theory, cyclic (co-) homology, K-homology

Goals: classical mathematical goals: index theorems (Connes, Moscovici) (generalized Atiyah-Singer)

Physics goals: description of topological quantization (best known example: IOHE) (quasicrystals, tilings) (index theorems)

Canonical quantization

What is *phase space* in classical mechanics? Newton: particles of mass m subject to forces F

$$(*) \qquad m\ddot{q} = F\,, \quad q : \underset{\text{time}}{\mathbb{R}} \to \underset{n-\text{space}}{\mathbb{R}^n}$$

$$F = F(q, \dot{q}, t) \in \mathbb{R}^n$$

$(*)$ is a 2^{nd} order differential equation; initial conditions $q(t_0) = x_0$ and $\dot{q}(t_0) = v_0$

Important class of examples.

Example. $F = -\nabla V$ (gradient of a *potential*), $V : \mathbb{R}^n \to \mathbb{R}$

Reduce 2^{nd} order to 1^{st} order by *doubling the variables*

$v = \dot{q}$ leads to a 1^{st} order differential equation

But $v = \dot{q}$ is not always the best choice

Maupertuis-Lagrange-Euler. The *stationary action principle.*

More fundamental than m and F is a function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (Lagrange function) which produces the equations of motion by the principle of *stationary action.* Given initial conditions $q(t_0) = x_0$, $q(t_1) = x_1$

$$\mathrm{S}[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\mathrm{d}t$$

(S is a function on curves which satisfy the initial conditions)

Equation of motion is obtained as follows:

$$\frac{\delta \mathrm{S}}{\delta q} = 0 \quad \Longleftrightarrow \quad \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}}_{\textit{Euler-Lagrange}}\ ,\ q = (q_1, q_2, \ldots, q_n) \qquad (**)$$

Example. $F = -\nabla V$, and $L(q, \dot{q}) = \frac{m}{2}\dot{q} \cdot \dot{q} - V(q)$. Then $(**) = (*)$.

"A classical physical system is described by a Lagrangian."

(Neglecting non-conservative systems with explicit time dependence: $L(q, \dot{q}, t)$)

(One can abstract and generalize to L on more general spaces)

Hamilton-Jacobi: reduce the 2^{nd} order differential equation $(**)$ to 1^{st} order by introducing the variables

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \text{ (canonical momentum variables)}$$

Example (continued)**.** $L(q, \dot{q}) = \frac{m}{2}\dot{q} \cdot \dot{q} - V(q) \implies p_i = m\dot{q}_i$

L is *regular* (generally):

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \text{ is an invertible matrix}$$

and $\dot{q}_i(p, q)$ exists (locally).

Hamilton function:

$$H(p, q) = \sum_i \dot{q}_i(p, q)p_i - L(q, \dot{q}(p, q))$$

for which

$$\frac{\partial H}{\partial q_j} = \sum_i \frac{\partial \dot{q}_i}{\partial q_j}p_i - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial q_j} = -\frac{\partial L}{\partial q_j} \underset{\text{by}(**)}{=} -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}$$

Hamilton-Jacobi equations:

$$(***) \qquad \frac{\partial H}{\partial q_j} = -\dot{p}_j\ ,\ \frac{\partial H}{\partial p_j} = \dot{q}_j \quad (1^{\text{st}} \text{ order equations})$$

Example (continued further)**.**

$$H(p, q) = \underbrace{\frac{p \cdot p}{2m} + V(q)}_{\mathrm{E_{kin}} + \mathrm{E_{pot}}} \quad \text{(energy function)}$$

(Conservative system: H is time independent)

This is *phase space*: the space of p's and q's (coordinates and conjugate momenta)

A commutative Fréchet algebra: $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$

($C_{\mathrm{b}}{}^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$ is a commutative C*-algebra, now over $\mathbb{C}$)

Semi-norms for $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$: for a multi-index α and a compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$:

$$S_{\alpha,K}(f) = \sup_{x \in K} |D^\alpha f(x)|$$

A *Poisson bracket*:

$$\{\cdot, \cdot\} : C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \times C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

is a Lie bracket:

$$\begin{aligned} \{f, g\} &= -\{g, f\} \\ \{f, \{g, h\}\} &= \{\{f, g\}, h\} + \{g, \{f, h\}\} \end{aligned}$$

and, for each $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\{f, \cdot\}$ is a derivation:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

As a consequence, $\{q_i, q_j\} = 0 = \{p_i, p_j\}$, $\{q_i, p_j\} = \delta_{ij}$ and

$$\begin{aligned} (***) \quad \Longleftrightarrow \quad & \dot{q} = \{q, H\}\,, \quad \dot{p} = \{p, H\}\,, \text{ and} \\ & \{f, H\} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = \dot{f}\,, \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \end{aligned}$$

So, H is the generator of the time evolution of any function $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

A conservative classical mechanical system is a Poisson algebra (commutative algebra of smooth functions over a manifold with Poisson bracket and a function H which generates the time evolution.)

States are points of the space.

2 (23 Jan. 2009)

Quantization

A classical mechanical (conservative) system is an algebra of smooth functions over a manifold with Poisson bracket and a Hamiltonian function H generating the time evolution with the help of the $\{\_,\_\}$.

$$(C^\infty(\mathcal{P}), \{\_,\_\}) \quad \mathcal{P} \text{ is phase space}$$

Quantization. Vaguely, a quantization of $(\mathcal{A}_0 = C^\infty(\mathcal{P}), \{\_,\_\}, |\_|)$ would be an (associative) algebra $(\mathcal{A}_\hbar, \hat{\mathcal{H}})$, where $\hbar$ is the minimal volume in phase space, such that

$$f \in \mathcal{A}_0 \longleftrightarrow f_\hbar \in \mathcal{A}_\hbar$$

satisfying

$$\frac{1}{i\hbar}[f_\hbar, g_\hbar] = \{f, g\}_\hbar + \mathcal{O}(\hbar)$$

Note: physicists prefer to think of $f_\hbar \mapsto f$ as $\hbar \to 0$

Deformation quantization: Deform $(\mathcal{A}_0, \cdot)$ into $(\mathcal{A}_0, *_\hbar)$, where $*_\hbar$ is the Moyal product; mostly done formally working with power series in $\hbar$: $(\mathcal{A}_0[[\hbar]], *_\hbar)$.

Strict quantization. (M. Rieffel, Dirac, N. P. Landsman)

Definition. A *C*-field* (*continuous field* of C*-algebras) over a locally compact Hausdorff space I is

$$\left(\mathcal{C}, (\mathcal{A}_\hbar)_{\hbar\in I}, (\phi_\hbar)_{\hbar \in I}\right)$$

where $\mathcal{C}$ is a C*-algebra, $\mathcal{A}_\hbar$ is a C*-algebra, $\phi_\hbar : \mathcal{C} \to \mathcal{A}_\hbar$ is a surjective C*-morphism, and

- F1) for $a \in \mathcal{C}$, $\|a\|_\mathcal{C} = \sup_{\hbar\in I}\|\phi_\hbar(a)\|_{\mathcal{A}_\hbar}$
- F2) for $a \in \mathcal{C}$, $(\hbar \mapsto \|\phi_\hbar(a)\|) \in C_0(I)$
- F3) $\mathcal{C}$ is a left $C_0(I)$-module ($f \in C_0(I)$, $a \in \mathcal{C} \Rightarrow fa \in \mathcal{C}$) such that $\phi_\hbar(fa) = f(\hbar)\phi_\hbar(a)$

(see Fig. 1) Think of $\mathcal{C}$ as the algebra of sections in the bundle case. A field of C*-algebras differs from a bundle in that the C*-algebra fibers are not necessarily all isomorphic.

In the first applications $I = [0, \hbar]$ or $I = [0, 1]$, and the field is indexed as follows

$$\left(\mathcal{C}, (\mathcal{A}_s)_{s\in I}, (\phi_s)_{s\in I}\right)$$

By definition, a *section* in the C*-field $(\mathcal{C}, (\mathcal{A}_s)_s, (\phi_s)_s)$ is a collection $(a_s \in \mathcal{A}_s)_s$ for which there exists a $c \in \mathcal{C}$ such that $\phi_s(c) = a_s$, for $s \in I$.

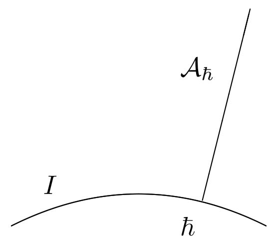


Figure 1: C*-field

Definition. (Rieffel) A *strict quantization* of $(C^\infty(\mathcal{P},\mathbb{C}),\{\_,\_\})$ (complexified Poisson algebra) is a C*-field $(\mathcal{C},\mathcal{A}_s,\phi_s)_{s\in I}$, $I=[0,\hbar]$, together with a dense subalgebra $\tilde{\mathcal{A}}_0$ of $\mathcal{A}_0 \stackrel{\text{def}}{=} C_0(\mathcal{P})$ on which $\{\_,\_\}$ is well-defined, and on which there is a linear map $Q:\tilde{\mathcal{A}}_0\to\mathcal{C}$ satisfying

i) $\phi_s\circ Q(f)|_{s=0}=f$, $f\in\tilde{\mathcal{A}}_0$, $s\in I$

ii) Q preserves $*$: $Q(f^*)=Q(f)^*$, $f\in\tilde{\mathcal{A}}_0$

iii) (most important)($Q_s \stackrel{\text{def}}{=} \phi_s\circ Q$)

$$\left\|\frac{1}{is}[Q_s(f),Q_s(g)]-Q_s(\{f,g\})\right\| \underset{s\to 0}{\longrightarrow} 0$$

$(f,g\in\tilde{\mathcal{A}}_0)$

From F2):

$$\left\|\frac{1}{2}\left(Q_s(f)Q_s(g)+Q_s(g)Q_s(f)\right)-Q_s(fg)\right\| \underset{s\to 0}{\longrightarrow} 0$$

Example. $\mathcal{P}=\mathbb{R}\times\mathbb{R}$, phase space for one variable; $(q,p)\in\mathcal{P}$ are the canonical variables: $\{p,q\}=0$. And

$$[Q_\hbar(p),Q_\hbar(q)]=i\hbar$$

which cannot be implemented using bounded linear operators on a separable Hilbert space. There cannot be a C*-algebra implementation.

Let $\hat{p}=Q_\hbar(p)$ and $\hat{q}=Q_\hbar(q)$. Introduce *Weyl operators* (unitary)

$$W_q(a)=\exp(ia\hat{q}) \quad W_p(b)=\exp(ib\hat{p}) \quad (a,b\in\mathbb{R})$$

$$\begin{aligned} W_q(a)W_p(b)W_q(a)^* &= W_p(b-a) \\ W_p(b)W_q(a)W_p(b)^* &= W_q(a+b) \end{aligned}$$

Proposal for $\mathcal{A}_\hbar$: C*-algebra generated by these relations. Details in the next lecture.

Theorem (Stone-von Neumann)**.** *Up to unitary equivalence there is only one irreducible, faithful representation of the Weyl algebra. It is given by the Schrödinger representation on* $L^2(\mathbb{R})$*:*

$$(\hat{q}\psi)(x) = x\psi(x) \quad (\hat{p}\psi)(x) = \frac{\hbar}{i}\psi(x)$$

where ψ *is restricted to some* core *of* $L^2(\mathbb{R})$*, for example the Schwartz space* $\mathcal{S}(\mathbb{R})$*.*

3 (26 Jan. 2009)

Quantization (cont.)

Strict quantization. (M. Rieffel, Dirac, N. P. Landsman)

Some notation from the previous lecture (see Fig. 2):

$$\underset{\text{quantizer}}{Q} : \mathcal{A}_0 \to \mathcal{C} \quad *\text{-linear map between C*-algebras}$$

$$Q_s \stackrel{\text{def}}{=} \phi_s \circ Q : \mathcal{A}_0 \to \mathcal{A}_s$$

$$\left\| \frac{1}{is}[Q_s(f), Q_s(g)] - Q_s(\{f,g\}) \right\| \underset{s\to 0}{\longrightarrow} 0$$

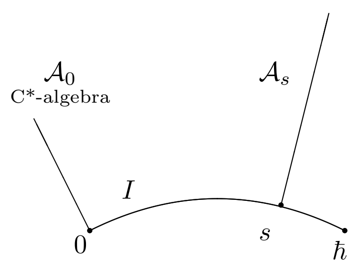


Figure 2: C*-field on $[0, \hbar]$

Example (Fresh Example: *Weyl quantization*; a mathematically precise version of Heisenberg quantization.). $\mathcal{P} = \mathbb{R}\times\mathbb{R}$, phase space for one variable; $(q,p) \in \mathcal{P}$ are the canonical variables.

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\,, f,g \in C^\infty(\mathbb{R}\times\mathbb{R})$$

This is given uniquely by $\{q,p\} = 1$ and $\{q,q\} = 0 = \{p,p\}$.

Heisenberg: it suffices to quantize $\{q,p\} = 1$ and $\{q,q\} = 0 = \{p,p\}$:

(CCR) $\qquad [Q_\hbar(q), Q_\hbar(p)] = i\hbar 1\ ,\ [Q_\hbar(q), Q_\hbar(q)] = 0\ ,\ [Q_\hbar(p), Q_\hbar(p)] = 0$

(*canonical commutation relations*, CCR). Set $\hat{q} = Q_\hbar(q)$ and $\hat{p} = Q_\hbar(p)$. Then $\hat{q}$ and $\hat{p}$ "should be" self-adjoint, but the (CCR) cannot be represented by bounded linear operators on a separable Hilbert space (previous exercise), and so $\hat{q}$ and $\hat{p}$ cannot live in a C*-algebra because C*-algebras have faithful representations.

Weyl: $(\hbar = 1)$

$$W_q(a) = \mathrm{e}^{ia\hat{q}} \quad W_p(b) = \mathrm{e}^{ib\hat{p}} \quad (a, b \in \mathbb{R})$$

$(CCR) \Rightarrow (WCCR)$:

(WCCR)
$$W_q(a)\hat{p}W_q(a)^{-1} = \hat{p} - a1$$
$$W_p(b)\hat{p}W_p(b)^{-1} = \hat{q} + b1$$

(Correction to equations given last time:

$$W_q(a)W_p(b)W_q(a)^{-1} = \mathrm{e}^{ib(\hat{p}-a1)} = W_p(b)\mathrm{e}^{-iab1} \quad \text{, etc.)}$$

Theorem (Stone-von Neumann)**.** *Up to unitary equivalence there is a unique irreducible representation for $\hat{p}$ and $\hat{q}$ on a separable Hilbert space such that the (WCCR) hold.*

This representation is the Schrödinger representation:

$$(\hat{q}\psi)(x) = x\psi(x) \quad (\hat{p}\psi)(x) = \frac{\hbar}{i}\psi'(x)$$

Suppose that $f, g \in \mathcal{S}(\mathbb{R})$ (Schwartz functions) ($\Rightarrow \hat{f}, \hat{g} \in \mathcal{S}(\check{\mathbb{R}})$ ($\check{\mathbb{R}} \cong \mathbb{R}$), where [Fourier] $\hat{f}(\xi) = (1/\sqrt{2\pi})\int \exp(-ix\xi)f(x)\mathrm{d}x$, etc.):

Definition.

$$W_q(f) = \frac{1}{\sqrt{2\pi}}\int \hat{f}(\xi)\mathrm{e}^{i\xi\hat{q}}\mathrm{d}\xi$$
$$W_p(g) = \frac{1}{\sqrt{2\pi}}\int \hat{g}(\xi)\mathrm{e}^{i\xi\hat{p}}\mathrm{d}\xi$$

$\hat{p}, \hat{q}$ in the Schrödinger representation.

Look at the subalgebra of $\mathcal{B}(L^2(\mathbb{R}))$ generated by $\{\, W_q(f)W_p(g) \mid f, g \in \mathcal{S}(\mathbb{R})\}$.

$$(W_q(f)\psi)(x) = \frac{1}{\sqrt{2\pi}}\int \hat{f}(\xi)\mathrm{e}^{i\xi x}\psi(x)\mathrm{d}\xi = f(x)\psi(x)$$
$$(W_p(g)\psi)(x) = \frac{1}{\sqrt{2\pi}}\int \hat{g}(\xi)\psi(x-\xi)\mathrm{d}\xi$$

$$(W_q(f)W_p(g)\psi)(x) = \int K_{xy}\psi(y)\mathrm{d}y$$
$$K_{xy} = \frac{1}{\sqrt{2\pi}}\hat{f}(x)\hat{g}(y-x) \in \mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$$

so $W_q(f)W_p(g)$ is Hilbert-Schmidt (hence compact). Taking the closure in the operator norm:

$$\overline{\{\, W_q(f)W_p(g) \mid f, g \in \mathcal{S}(\mathbb{R})\}} = \mathcal{K}(L^2(\mathbb{R})) \subset \mathcal{B}(L^2(\mathbb{R}))$$

where $\mathcal{K}(L^2(\mathbb{R}))$ is the algebra of compact operators.

Will show that this algebra is a cross-product with $\mathbb{R}$.

Claim. $\mathcal{K}(L^2(\mathbb{R})) \cong C_0(\mathbb{R}) \rtimes_\alpha \check{\mathbb{R}}$, where α is the action of $\check{\mathbb{R}}$ on $\mathbb{R}$ induced by translation: $\alpha : \check{\mathbb{R}} \to \mathrm{Aut}(C_0(\mathbb{R}))$; $(\alpha_\xi(f))(x) = f(x-\xi)$, $f \in \check{\mathbb{R}}$.

Recall the definition of $\mathcal{B} \rtimes_\alpha \mathbb{R}$, $\mathcal{B}$ a C*-algebra, $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{B})$ a continuous action. Two steps.

1) L^1-crossed product: $L^1(\mathbb{R}, \mathcal{B})$ completion of $\mathcal{S}(\mathbb{R}, \mathcal{B})$ with respect to

$$\|f\|_1 = \int \|f(x)\|_{\mathcal{B}} \mathrm{d}x$$

(See Lang's *Analysis* for definitions in terms of step functions $\mathbb{R} \to \mathcal{B}$.) Introduce a product

$$(fg)(x) = \int f(y)\alpha_y(g(x-y))\mathrm{d}y$$

and a $*$-structure

$$f^*(x) = \alpha_x(f(-x)^*)$$

2) Universal C*-closure:

$$\|f\| = \sup_\pi \|\pi(f)\|_{\mathcal{B}(\mathcal{H})}\,, \quad \text{(sup over bounded representations } \pi \text{ of } L^1(\mathbb{R},\mathcal{B}) \text{ on } \mathcal{H})$$

Theorem. *This* sup *is attained in a certain representation.*

Let $(\phi, \mathcal{H})$ be a representation of $\mathcal{B}$. It induces a representation on $L^2(\mathbb{R}, \mathcal{H})$ which is faithful if ϕ is:

$$(\pi(f)\psi)(x) = \int \phi(\alpha_{-x}(f(x-y)))\psi(y)\mathrm{d}y$$

Apply this to $\mathrm{ev}_0 : C_0(\mathbb{R}) \to \mathbb{C}$.

Proposition. *The induced representation is faithful (and the* sup *is attained in that representation).*

π constructed with ev_0: $\pi(C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R}) = \mathcal{K}(L^2(\mathbb{R}))$.

The integral kernel of $\pi(f)$: $(\pi(f))_{xy} = f(x-y)(x)$, $f : \mathbb{R} \to C_0(\mathbb{R})$. If

$$f \in \mathcal{S}(\mathbb{R}, C_0(\mathbb{R})) \overset{\text{dense}}{\subset} L^1(\mathbb{R}, C_0(\mathbb{R}))$$

then this is L^2 hence $\pi(f)$ is Hilbert-Schmidt.

Conclusion:

$$C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \cong \mathcal{K}(L^2(\mathbb{R}))$$

Now can construct total algebra.

4 (28 Jan. 2009)

Quantization (cont.)

From previous lecture:

$$\mathcal{A}_s \stackrel{\text{def}}{=} \underset{q}{C_0(\mathbb{R})} \rtimes_{\alpha^s} \underset{\xi}{\check{\mathbb{R}}},\ \text{where } (\alpha^s_\xi(f))(q) = f(q - s\xi), s > 0,\ \text{and } \mathcal{A}_0 = C_0(\mathbb{R})$$

$\mathcal{A}_s \cong \mathcal{K}(L^2(\check{\mathbb{R}}))\,,\ \forall s > 0.$

Dynamical system $(\mathbb{R}, \mathbb{R}, \tilde{\alpha}^s)$, with $\tilde{\alpha}^s_\xi(q) = q - s\xi$

Want to erase all these details: go to the group algebra of the Heisenberg group; arranging $\{\mathcal{A}_s\}_{s\in\mathbb{R}}$ into a C*-field.

$$\underset{\text{Heisenberg group}}{\mathbb{H}} = \left\{ \begin{pmatrix} 1 & \xi & q \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \xi, q, s \in \mathbb{R} \right\} \subset \mathrm{SL}(3, \mathbb{R})$$

Subgroup $\mathbb{R}^2 \subset \mathbb{H}$:

$$\mathbb{R}^2 \cong \left\{ \begin{pmatrix} 1 & 0 & q \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| \ q, s \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 1 & 0 & q_1 \\ 0 & 1 & s_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & q_2 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & q_1 + q_2 \\ 0 & 1 & s_1 + s_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{H}$ is non-commutative:

$$\begin{pmatrix} 1 & \xi_1 & q_1 \\ 0 & 1 & s_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi_2 & q_2 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi_1 + \xi_2 & q_1 + q_2 + \xi_1 s_2 \\ 0 & 1 & s_1 + s_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{H} \cong \mathbb{R}^2 \rtimes_{\tilde{\tau}} \mathbb{R}$ (semi-direct product), with $\tilde{\tau}_\xi(q, s) = (q - s\xi, s)$.

The group algebra of the Heisenberg group, $C^*\mathbb{H}$, in terms of the group algebra $C^*(\mathbb{R}^2)$, is

$$C^*\mathbb{H} = C^*(\mathbb{R}^2) \rtimes_\tau \mathbb{R} = \underbrace{C_0(\mathbb{R}^2) \rtimes_\tau \mathbb{R}}_{\text{our total algebra } \mathcal{C}}\,,\ (\tau_\xi f)(q, s) = f(q - s\xi, s)$$

Must define morphisms from $\mathcal{C}$ to fibers:

$$\phi_s : \mathcal{C} \to \mathcal{A}_s = C_0(\mathbb{R}) \rtimes_{\alpha^s} \check{\mathbb{R}}$$

(s plays the role of Planck's constant).

$$\mathrm{ev}_s : C_0(\mathbb{R}^2) \to C_0(\mathbb{R})\,, \quad (\mathrm{ev}_s f)(q) = f(q, s)$$

(Leaves of the τ-action are $\mathbb{R} \times \{s\}$ and $\{q\} \times \{0\}$.)

So

$$\phi_s : C_0(\mathbb{R}^2) \rtimes_\tau \mathbb{R} \to C_0(\mathbb{R}) \rtimes_{\alpha^s} \mathbb{R}$$

is given by

$$(\phi_s(F)(\xi))(q) := F(\xi)(q,s)\,, \quad F : \mathbb{R} \to C_0(\mathbb{R}^2)$$

Claim. $(\mathcal{C}, (\mathcal{A}_s, \phi_s)_{s\in\mathbb{R}})$ is a C*-field. (Proof is an exercise.)

That's the first step. Now want to see that this is a strict quantization.

Proposition. $\left(\mathcal{C}|_{[0,\hbar]}\,, \{\mathcal{A}_s, \phi_s\}_{s\in[0,\hbar]}\right)$ *is a strict quantization of* $\left(C_0^\infty(\underset{q}{\mathbb{R}} \times \underset{p}{\mathbb{R}}), \{\_\,,\,\_\}\right)$

Proof.

$$\begin{aligned}\mathcal{A}_0 &= \underbrace{C_0(\underset{q}{\mathbb{R}}) \rtimes_{\alpha^0} \underset{\xi}{\mathbb{R}}}_{\text{action is trivial}} \cong C_0(\mathbf{R}) \underset{\text{group algebra of } \check{\mathbb{R}}}{\otimes\quad C^*(\check{\mathbb{R}})} \\ &\underset{\text{id}\otimes\text{FT}}{\cong} C_0(\underset{q}{\mathbb{R}}) \otimes C_0(\underset{p}{\mathbb{R}}) \\ &\cong C_0(\mathbb{R}^2) \underset{\text{densely}}{\supset} C_0^\infty(\mathbb{R}^2)\end{aligned}$$

Next need quantization map $Q : \mathcal{A}_0 \to \mathcal{C}$

$$(\phi_s \circ Q(f))(\xi)(q) = \underset{C_0(\mathbb{R})\rtimes_{\alpha^0}\check{\mathbb{R}}}{f(\xi)q}$$

Long calculation follows.

$$\left\| \frac{1}{is}\left[Q_s(f), Q_s(g)\right] - Q_s\left(\{f,g\}\right) \right\| \xrightarrow{s\to 0} 0$$

$$\begin{aligned}[Q_s(f), Q_s(g)](\xi) &\overset{\text{def'n of product}}{=} \int [f(\eta)\alpha^s_\eta g(\underbrace{\xi-\eta}_{\eta'}) - g(\eta)\alpha^s_\eta f(\xi-\eta)]\mathrm{d}\eta \\ &\overset{\text{change variable}}{=} -\int [f(\xi-\eta')\alpha^s_{\xi-\eta'} g(\eta') - g(\xi-\eta')\alpha^s_{\xi-\eta'} f(\eta')]\mathrm{d}\eta' \\ &\overset{\eta \text{ for } \eta'}{=} -\int [f(\xi-\eta)\alpha^s_{\xi-\eta} g(\eta) - g(\xi-\eta)\alpha^s_{\xi-\eta} f(\eta)]\mathrm{d}\eta\end{aligned}$$

Evaluate at q:

$$
\begin{aligned}
[Q_s(f), Q_s(g)](\xi)(q) &= -\int [f(\xi-\eta)(q)g(\eta)(q-s(\xi-\eta)) - f(\xi-\eta)(q-s\eta)g(\eta)(q)]\mathrm{d}\eta \\
&= \int \Bigg[f(\xi-\eta)(q)g(\eta)(q-s(\xi-\eta)) - \underbrace{f(\xi-\eta)(q-s\eta)g(\eta)(q-s(\xi-\eta))}_{\text{add and subtract}} \\
&\qquad + \underbrace{f(\xi-\eta)(q-s\eta)g(\eta)(q-s(\xi-\eta))}_{\text{add and subtract}} - f(\xi-\eta)(q-s\eta)g(\eta)(q) \Bigg] \mathrm{d}\eta \\
&\overset{\text{MVT}}{=} \int \Bigg[\frac{\partial f(\xi-\eta)}{\partial q}(\dot{q})(-s\eta)g(\eta)(q-s(\xi-\eta)) \\
&\qquad - \frac{\partial g(\xi-\eta)}{\partial q}(\dot{q}')(-s\eta)f(\eta)(q-s(\xi-\eta)) \Bigg] \mathrm{d}\eta \quad (\dot{q}, \dot{q}' \in \langle q, q-s\eta \rangle) \\
&\Bigg(\text{Divide by } is \text{ to cancel some } s\text{'s; use } i\eta g(\eta) \underset{\text{FT}}{\cong} \frac{\partial \hat{g}}{\partial p}(p) \\
&\qquad \text{boundary vanishes; convolution product replaced by product} \Bigg) \\
&= \frac{\partial \hat{f}}{\partial q}(\dot{q}) \frac{\partial \hat{g}}{\partial p}(p)(\dot{q}') - \frac{\partial \hat{g}}{\partial q}(\dot{q}') \frac{\partial f}{\partial p}(\dot{q}) \xrightarrow{\dot{q}, \dot{q}' \to q \text{ as } s \to 0} \{\hat{f}, \hat{g}\}
\end{aligned}
$$

The remainder of the proof is an exercise. □

5 (30 Jan. 2009)

Mostly about physics

Summary: (strict quantization $\Rightarrow$ QM mainly non-commutative) The *fundamental proposition of Heisenberg*:

> *Observables* from a non-commutative, associative algebra; in particular "there are elements $\hat{p}, \hat{q}$" which satisfy ($\hat{}$ for possibly unbounded)
>
> $$[\hat{p}, \hat{q}] = -i\hbar 1$$
>
> Furthermore there is an element $\hat{H}$ such that the time evolution of an observable A is given by
>
> $$\dot{A} = \frac{1}{i\hbar}[A, \hat{H}]$$

Definition. An *abstract QM system* is a $*$-algebra (algebra with involution) $\mathcal{A}$ together with a Hamiltonian H *affiliated to* $\mathcal{A}$ (see below) which governs the evolution of A via the equation

$$(*) \qquad \dot{A} = \frac{1}{i\hbar}[A, H]$$

For topology $\mathcal{A}$ should be a Banach algebra (or C*-algebra, or von Neumann algebra) which is separable.

In this context H *is affiliated to* $\mathcal{A}$ if for every bounded continuous function f (possibly vanishing at infinity), $f(H) \in \mathcal{A}$.

Integrated form of $(*)$

$$\begin{aligned} A(t) &= \mathrm{Ad}_{\mathcal{U}(t,t_0)} A(t_0) = \mathcal{U}(t,t_0) A(t_0) \mathcal{U}(t,t_0)^{-1} \\ \mathcal{U}(t,t_0) &= \mathrm{e}^{i(t-t_0)H} \end{aligned}$$

(conservative in that H is independent of t)

Heisenberg picture of QM (evolution of elements of the algebra). A *concrete* QM system is a representation of an abstract QM system. So there is also a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space.

WLOG π is given by a GNS representation w.r.t. a state on $\mathcal{A}$. Recall that a *state* ϕ on $\mathcal{A}$ is a positive, normalized continous functional on $\mathcal{A}$:

$$\begin{gathered} \phi \in \mathcal{L}(\mathcal{A}, \mathbb{C}) \text{ (continuous linear functional)} \\ \forall A \in \mathcal{A} : (A \text{ positive} \Rightarrow \phi(A) \in \mathbb{R}^+) \text{ (positive functional)} \\ \phi(1) = 1 \text{ (unital algebra), or } \lim_\lambda \phi(E_\lambda) = 1 \text{ (approx. id.) (normalized)} \end{gathered}$$

The dynamical evolution of $\mathcal{A}$ corresponds to a *dual* evolution on $\mathcal{L}(\mathcal{A}, \mathbb{C})$:

$$(**) \qquad \phi(t) = \phi(t_0) \circ \mathrm{Ad}_{\mathcal{U}(t,t_0)}$$

$$\mathcal{U}_{t,t_0} \text{ unitary} \Rightarrow \mathrm{Ad}_{\mathcal{U}(t,t_0)} \text{ is a positive map}$$

$$\mathrm{Ad}_{\mathcal{U}(t,t_0)} 1 = 1$$

Hence the *state space* $\mathcal{S}\mathcal{A}$ is invariant under this evolution. The *abstract Schrödinger picture* is $(**)$ restricted to $\mathcal{S}\mathcal{A}$. The state space $\mathcal{S}\mathcal{A}$ is a convex set whose extreme points are called *pure states.*

Theorem (Proof not given here)**.** *The GNS representation of $\phi \in \mathcal{S}\mathcal{A}$ is irreducible $\iff$ ϕ is pure.*

Example. $\mathcal{A} = M_2(\mathbb{C})$; $\phi = \mathrm{Tr}$ is a pure state (a positive matrix has positive eigenvalues); ¿ $M_2(\mathbb{C}) = \mathcal{B}(\mathbb{C}^2)$?

Let $\psi \in \mathbb{C}^2$ with $\|\psi\| = 1$; $\phi = \langle\psi|\ |\psi\rangle$, so $\phi(A) = \langle\psi, A\psi\rangle = \langle c\psi, Ac\psi\rangle$, $c \in \mathbb{C}$, $|c| = 1$ (to change ψ by phase is to multiply by a complex number of unit modulus).

To be specific,

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_1 = \langle\psi_1|\ |\psi_1\rangle$$

and

$$\mathcal{H}_{\phi_1} = \overline{\mathcal{A}/\mathcal{N}_{\phi_1}}\,, \text{ where } \mathcal{N}_{\phi_1} = \{ A \in \mathcal{A} \mid \phi_1(A^*A) = 0 \}$$

Since $\phi_1(A^*A)$ is the $(1,1)$-element of A^*A, this means that

$$A \in \mathcal{N}_{\phi_1} \quad \iff \quad \sum_i \bar{A}_{i\,1} A_{i\,1} = 0 \quad \iff \quad A_{i\,1} = 0,\ i = 1, 2$$

so

$$\mathcal{N}_{\phi_1} = \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\} \quad \Longrightarrow \quad \mathcal{A}/\mathcal{N}_{\phi_1} \cong \left\{ \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \right\} \cong \mathbb{C}^2$$

This representation is clearly irreducible (exercise).

Now the other case.

$$\mathcal{N}_{\phi_1} = \{ A \mid \mathrm{Tr}(A^*A) = 0 \} = \{ A \mid A^*A = 0 \} = \{ A \mid A = 0 \}$$

so

$$\mathcal{A}/\mathcal{N}_{\phi_1} \cong \mathcal{A}$$

which cannot be irreducible since $\mathcal{H}_{\phi_1}$ and $\mathcal{A}$ have the same dimension.

Any state is a linear combination of pure states and evolution is linear so the dual evolution is determined on the set of pure states.

Concrete Schrödinger picture: π is an irreducible representation.

Theorem. *The pure states of* $\pi(\mathcal{A})$ *are* vector states*: to* ϕ*, pure on* $\pi(\mathcal{A}) \leftrightarrow \psi \in \mathcal{H}\,, \|\psi\| = 1\,:\, \phi = \langle\psi|\ |\psi\rangle$.

Look at the evolution of ψ.

$$(**) \qquad \underset{\leftrightarrow\psi(t)}{\phi(t)} = \underset{\leftrightarrow\psi}{\phi(t_0)} \circ \mathrm{Ad}_{\mathcal{U}(t,t_0)} = \left\langle \psi \,\middle|\, \mathcal{U}_{t,t_0} \cdot \mathcal{U}^*_{t,t_0} \middle|\, \psi \right\rangle = \left\langle \mathcal{U}^*_{t,t_0}\psi \,|\,|\, \mathcal{U}^*_{t,t_0}\psi \right\rangle$$

So, the integrated Schrödinger equation:

$$\psi(t) = \mathcal{U}^*_{t,t_0}\psi(t_0) = \mathrm{e}^{-i(t-t_0)H/\hbar}\psi(t_0) \quad (\psi(t_0) = \psi)$$

Differentiated form of the Schrödinger equation:

$$\dot{\psi} = \frac{1}{i\hbar}H\psi$$

(The eigenvalue equation for H is the stationary Schrödinger equation.)

What is *measurement*? A single *object* (particle) is a pure state ϕ on the algebra (or use GNS vector in $\mathcal{H}$). Experiment measures observable A of the state (a self-adjoint element of the algebra). In the classical case $\phi(A) \in \mathbb{R}$ is the result of the measurement (ϕ is a point). In QM we only get a probability distribution: a single measurement yields a random result.

To A corresponds a spectrum $\sigma(A) \subset \mathbb{R}$. Let C be a measurable (Borel) subset of $\sigma(A)$. $\chi_C(A)$ (the characteristic function χ_C applied to A, as defined by the spectral calculus) is actually the spectral projection onto the spectral subspace of A corresponding to C, and (axiom) $\phi(\chi_C(A))$ is the probability that object ϕ has value of the observable A in C.

6 (2 February 2009)

Measurement (cont. from previous lecture)

Device detects value of observable in $C \overset{\text{Borel}}{\subset} \sigma(A)$, $A = A^* \in \pi(\mathcal{A})$ (π is a representation); particle is $\phi \in \mathcal{PS}(\pi(A))$ (pure states).

$\chi_C(A)$ = spectral projection of $\pi(A)$ onto spectral values in C

$\phi(\chi_C(A))$ = probability of measuring value for A which belongs to C

In general

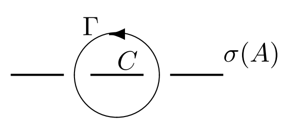
$$\chi_C(A) = \int_C \mathrm{d}E_\lambda \in \overline{\pi(A)}^{\mathrm{WOT}}, \text{ where } A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda$$

"WOT" denotes the weak operator topology; $\overline{\pi(A)}^{\mathrm{WOT}}$ is a von Neumann algebra which is typically a non-separable C*-algebra; $\{E_\lambda\}_\lambda$ is a projection-valued measure.

If C is isolated in the spectrum of A ($\sigma(A) \subset \mathbb{R}$), then

$$\chi_C(A) = \frac{1}{2\pi i} \oint_\Gamma \frac{1}{H - z} \mathrm{d}z$$

where Γ is a suitable plane contour separating C from $\sigma(A) \setminus C$. Alternatively,

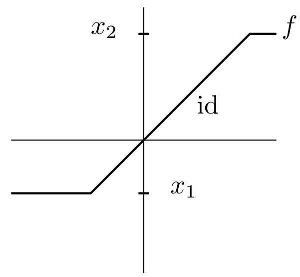


$\chi_C(A) = \tilde{\chi}_C(A)$ where $\tilde{\chi}_C : \mathbb{R} \to \mathbb{R}$ is a continuous approximation of the indicator function of C such that

$$\tilde{\chi}_C(\lambda) = \begin{cases} 1 & \text{if } \lambda \in C \\ 0 & \text{if } \lambda \in \sigma(A) \setminus C \\ \text{whatever} & \text{elsewhere} \end{cases}$$

Example. $A = \hat{q}$; rather, $f(\hat{q})$ where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which is the identity on the interval $[x_1, x_2]$ and is constant on each of $(-\infty, x_1]$ and $[x_1, \infty)$. The operator $\hat{q}$ need not be bounded; the cutoff version $f(\hat{q})$ is bounded:

$$\begin{aligned} \mathrm{Spec}(\hat{q}) &= \mathbb{R} \\ \mathrm{Spec}(f(\hat{q})) &= [x_1, x_2] \supset C = [y_1, y_2] \\ \phi(\chi_C(f(\hat{q}))) &= \phi(\chi_C(\hat{q})) \end{aligned}$$



Suppose π is the Schrödinger representation on $L^2(\mathbb{R})$. Then $\exists \psi \in L^2(\mathbb{R})$; $\|\psi\| = 1$ such that $\phi(A) = \langle \psi \mid \underbrace{\mathbf{1}_C \psi}_{\text{indicator}} \rangle = \int_C \bar{\psi}\psi$.

> In other words, in the Schrödinger representation the absolute value squared of the wave function ψ is the probability density of the distribution measured for the operator $\hat{q}$.

Repeat the *same* experiment (same A, C) with the *same* state. Then we can measure the probability distribution. If we cannot repeat with the same state, then we need to incorporate the probability distribution for the state. The probability distribution then becomes

$$m_C(A) := \int \phi(\chi_C(A)) \mathrm{d}\nu(\phi)$$

where ν is a probability measure on $\mathcal{PS}\pi(A)$. $m_C(A) =$ the probability that the value of A is measured to be in C.

What is ν on $\mathcal{PS}\pi(A)$? Suppose that $\pi(H)$ (H generates the evolution) has discrete spectrum, i.e., $\sigma(A) = \{\text{eigenvalues of } H\}$ (e.g, harmonic oscillator). (λ is an eigenvalue if $\exists \psi \in \mathcal{H} \setminus \{0\}$, $\mathcal{H}$ is the Hilbert space of π, such that $H\psi = \lambda\psi$). Let $\{\psi_n \mid n \in \mathbb{N}\}$ be an ONB for $\mathcal{H}$ consisting of eigenvectors. Assign to each ψ_n the probability

$$Z^{-1}\mathrm{e}^{-\beta\lambda_n}$$

where $Z = \text{ normalizer } = \sum_n \mathrm{e}^{-\beta\lambda_n}$.

$$\begin{aligned}
\phi_n(A) &= \langle \psi_n \mid \pi(A)\psi_n \rangle \\
m_C(A) &= \sum_n \phi_n(A) Z^{-1} \mathrm{e}^{-\beta\lambda_n} \\
&= \frac{1}{Z} \sum_n \langle \psi_n | \mathrm{e}^{-\beta H} | \psi_n \rangle \\
&= \frac{1}{Z} \operatorname{Tr}\left(\mathrm{e}^{-\beta H}\right)
\end{aligned}$$

Here we may forget about the discrete spectrum: this trace formula holds for general s.a. H and relates to *trace class operators*. If A and B are trace class operators, then

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \text{ and } \mathrm{Tr}([A,B]) = 0$$

The *expectation value* for A is then

$$\begin{aligned}\frac{1}{Z}\mathrm{Tr}\left(\pi(A)\mathrm{e}^{-\beta H}\right) &= \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\pi(A)\right)\\ &= \mathrm{Tr}(\rho A), \text{ where } \rho = \frac{\mathrm{e}^{-\beta H}}{Z} \text{ (suppressing the rep'n)}\end{aligned}$$

Properties of ρ are

$$\begin{aligned}\text{i)} \quad & \rho \geq 0\\ \text{ii)} \quad & \mathrm{Tr}(\rho) = 1\end{aligned}$$

One-Particle Approximation for a Solid

Each electron feels an effective potential (electron-electron interaction is very well described by a *mean field*.) Measuring A gives the value of $\mathrm{Tr}(\rho A)$, where ρ is the 1-particle density matrix.

Later: the quantum Hall effect versus the fractional quantum Hall effect. Plan: show that $\mathrm{Tr}(\rho A)$ comes up in quantum topological invariants.

7 (4 February 2009)

Observable C*-algebras for systems described by Delone sets.

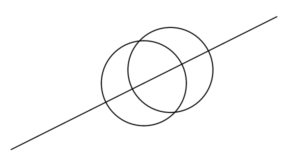
Delone set = set of (equilibrium) positions of the atoms in a solid or molecule. $\mathcal{P} \subset \mathbb{R}^n$. $B_r(x)$ = closed r-ball of $x \in \mathbb{R}^n$. $\mathcal{P} \subset \mathbb{R}^n$ then $\mathcal{P} - x = \{p - x \mid p \in \mathcal{P}\}$.

$B_r[\mathcal{P}] := B_r(0) \cap \mathcal{P}$ is the *r-patch* of $\mathcal{P}$ at 0 or *window* of $\mathcal{P}$ at 0 of radius r.

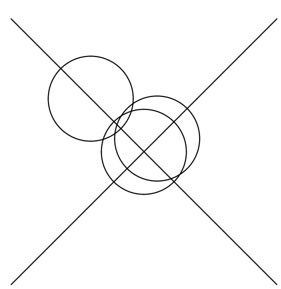
Definition. $\mathcal{P}$ has *finite local complexity* (FLC): $\Leftrightarrow \forall r : \{ B_r[\mathcal{P} - p] \mid p \in \mathcal{P}\}$ has finitely many elements.

Examples.

1) $\mathcal{P} = \{p\}$; then $B_r[\mathcal{P} - p]$ is the only possibility.
2) $\mathcal{P}$ is finite.
3) $\mathcal{P}$ is a lattice in $\mathbb{R}^n$.
4) $\mathcal{P}$ is an affine line in $\mathbb{R}^2$.



5) $\mathcal{P}$ is a quasi-periodic set.
6) (Counter example)



Two transverse affine lines in $\mathbb{R}^2$
and several windows

We allow $\mathcal{P}$ to be colored (finitely): equality requires same colors; equivalently, $\mathcal{P} \subset \mathbb{R}^n \times \{\text{colors}\}$.

Definition. A function $f : \mathbb{R}^n \to Y$ (set) is *r-$\mathcal{P}$-equivariant* if

$$B_r[\mathcal{P} - x] = B_r[\mathcal{P} - y] \Longrightarrow f(x) = f(y) \quad (x, y \in \mathbb{R}^n)$$

The value of f at x is determined by what is in the window. In particular $f|_{\mathcal{P}}$ has finite image.

Definition. f is *strongly* $\mathcal{P}$-equivariant (s-$\mathcal{P}$-e) if $\exists r > 0$ such that f is r-$\mathcal{P}$-e

Suppose that (Y, d) is metric. We look at continuous functions $\mathbb{R}^n \to Y$.

E.g., D is a metric on $C_b(\mathbb{R}^n, Y)$; $D(f, g) = \sup_x d(f(x), g(x))$:

$$\underset{weakly\text{-}\mathcal{P}\text{-e}}{C_{\text{w-}\mathcal{P}}}(\mathbb{R}^n, Y) := \overline{C_{\text{s-}\mathcal{P}}(\mathbb{R}^n, Y)}^D$$

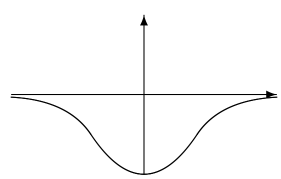
We are interested in $Y = \mathbb{C}$: $C_{\mathcal{P}}(\mathbb{R}^n) := C_{\text{w-}\mathcal{P}}(\mathbb{R}^n, \mathbb{C})$ is a separable C*-algebra.

Ex 1:

Suppose $\mathcal{P}$ is uniformly discrete (there is a minimum distance between points), and let $v : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp} v \subset B_R(0)$.

$$V : \mathbb{R}^n \to \mathbb{R} = \delta_{\mathcal{P}} * v\,, \qquad \underset{A^{\text{Borel}} \subset \mathbb{R}^n}{\delta_{\mathcal{P}}(A)} = |A \cap \mathcal{P}|$$

$$V(x) = \int \delta_{\mathcal{P}}(y) v(x - y) \mathrm{d}y = \sum_{p \in \mathcal{P}} v(x - p)$$



Short range atomic potential at each atom

$$\begin{aligned} V(x) = \int \delta_{\mathcal{P}}(y) v(x - y) \mathrm{d}y &= \sum_{p \in \mathcal{P}} v(x - p) \\ &= \sum_{p \in \mathcal{P} : x - p \in B_R(0)} v(x - p) = \sum_{y \in B_R[\mathcal{P} - x]} v(-y) \end{aligned}$$

so V is a w-$\mathcal{P}$-e function.

A potential resulting from a local atomic potential is a w-$\mathcal{P}$-e function. In general, coloring for different species of atoms: several v_{color}.

Ex 2:

$\mathcal{Q} \subset \mathbb{R}^n$ is *locally derivable* from $\mathcal{P}$ if $\mathbf{1}_\mathcal{Q}$ is s-$\mathcal{P}$-e: can construct $\mathcal{Q}$ by looking at windows in $\mathcal{P}$. Baake *et. al.* defined *mutually locally derivable* (MLD) which can be restated by saying that

$$\mathcal{Q} \text{ MLD } \mathcal{P} \iff \mathbf{1}_\mathcal{Q} \text{ is s-}\mathcal{P}\text{-e and } \mathbf{1}_\mathcal{P} \text{ is s-}\mathcal{Q}\text{-e}$$

Ex 3:

$\mathcal{P} = \{p\}$: f is R-$\mathcal{P}$-e if $|x-p| > R \Rightarrow f(x) = c = \text{constant}$. So f is R-$\mathcal{P}$-e $\Leftrightarrow \operatorname{supp}(f-c) \subset B_R(p)$.

$$\begin{aligned} C_{\text{s-}\mathcal{P}\text{-e}}(\mathbb{R}^n, \mathbb{C}) &= C_c(\mathbb{R}^n) + \mathbb{C} \\ C_{\text{w-}\mathcal{P}\text{-e}}(\mathbb{R}^n, \mathbb{C}) &= C_\mathcal{P}(\mathbb{R}^n) = C_0(\mathbb{R}^n) + \mathbb{C} \cong S^n \end{aligned}$$

Ex 3′:

$\mathcal{P}$ is finite (molecule) same as $\mathcal{P} = \{p\}$, mostly.

Ex 4:

$\mathcal{P}$ a compact lattice in $\mathbb{R}^n$: if R is large enough, any R-$\mathcal{P}$-e function is $\mathcal{P}$-periodic. So

$$\begin{aligned} C_{\text{s-}\mathcal{P}}(\mathbb{R}^n) = C_{\text{w-}\mathcal{P}}(\mathbb{R}^n) &= \text{ continuous periodic functions on } \mathbb{R}^n \\ &\cong C(\mathbb{R}^n/\mathcal{P}) \end{aligned}$$

$1 \in C_\mathcal{P}(\mathbb{R}^n)$ so $C_\mathcal{P}(\mathbb{R}^n)$ is a C*-algebra. Then Gelfand-Naimark: $\exists$ a compact Hausdorff space $\Omega_\mathcal{P}$ such that $C(\Omega_\mathcal{P}) \cong C_\mathcal{P}(\mathbb{R}^n)$.

Theorem. $\Omega_\mathcal{P} = \overline{\{\mathcal{P} - x\}}^{D_T}$

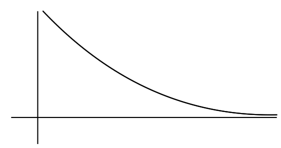
$$D_T(\mathcal{Q}, \mathcal{P}) := \sup \left\{ \frac{1}{R+1} \;\middle|\; B_R[\mathcal{P}] \underset{R}{\approx} B_R[\mathcal{Q}] \right\}, \text{ where}$$

$$B_R[\mathcal{P}] \underset{R}{\approx} B_R[\mathcal{Q}] \Leftrightarrow \exists_{x,y} \ \|x\|, \|y\| \le \frac{1}{R} : B_R[\mathcal{P} - x] = B_R[\mathcal{Q} - x]$$

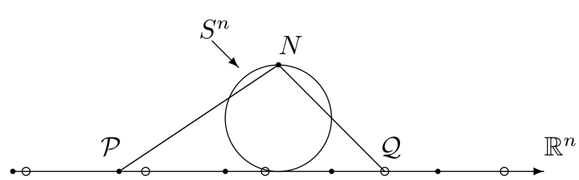
May replace $\frac{1}{R+1}$ with any $f(R)$ like this:

If we use $f(R) = \frac{1}{R}$, then $D_T(\mathcal{P} - x, \mathcal{P}) \cong \|x\|$ if x is small.

Alternatively, by a stereographic projection, $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\} \cong S^n$; compare the two sets in S^n using the Hausdorff metric:



$f(R)$



Stereographic projections of $\mathcal{P}$ and $\mathcal{Q}$

8 (6 February 2009)

Spectrum of $C_{\mathcal{P}}(\mathbb{R}^n)$

$\operatorname{Spec}(C_{\mathcal{P}}(\mathbb{R}^n)) =\ ?$

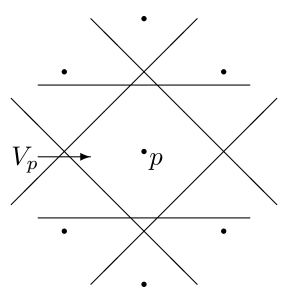
1) hull of $\mathcal{P} = \Omega_{\mathcal{P}} = \overline{\{\mathcal{P} - x\}}^{D_T}$
2) $\Omega_{\mathcal{P}} = \varprojlim (\Gamma_k, \alpha_k)$
3) $\Omega_{\mathcal{P}} \cong \operatorname{Spec}(C_{\mathcal{P}}(\mathbb{R}^n))$

1) $\Leftrightarrow$ 2) is known to the Bozeman audience.

The proof of 3) is very nice.

$\mathcal{P}$ FLC Delone $\rightsquigarrow$ Voronoi tiling $\mathcal{T}$: $\forall p \in \mathcal{P}$ Voronoi domain:

$$V_p = \{\, x \in \mathbb{R}^n \mid \|x - p\| \le \|x - p'\| \quad \forall p' \in \mathbb{R}^n \}$$



Voronoi domain V_p

- tile in $\mathcal{T}$ is a Voronoi domain.
- *k-neighborhood* of a tile $t =$
  {all tiles which meet $k-1$-neighborhood of t};
  where 0-neighborhood of a tile is the tile itself.

  The *geometric support* of a k-decorated tile is the support of the tile.
- *k-decorated* tile is the tile labelled with its k-neighborhood.
- k-decorated *prototile* is the equivalence class of a k-decorated tile under translation.

FLC $\Rightarrow$ only finitely many k-decorated prototiles. Define [F. Gähler]

$$\Gamma_k := \dot{\bigsqcup} (\text{ support of } k\text{-decorated prototiles}) \,/ \sim$$

Here, if f_1 and f_2 are faces of t_1 and t_2, resp., where t_1 and t_2 are k-decorated prototiles such that in $\mathcal{T}$ $\exists$ representatives $\tilde{t}_1$ and $\tilde{t}_2$ of t_1 and t_2 whose faces corresponding to f_1 and f_2 coincide, then f_1 is identified with f_2.

Important map: $\alpha_k : \Gamma_{k+1} \to \Gamma_k$, with $\alpha_k(x) = x$ the k-decorated prototile obtained by forgetting the $k+1$-collar (the tiles in the $k+1$-decoration which do not belong to the k-decoration). This map is obviously surjective.

Definition (for our purposes)**.**

$$\underset{(=\Omega_{\mathcal{T}})}{\Omega_{\mathcal{P}}} = \lim_{\leftarrow} (\Gamma_k, \alpha_k)$$

Direct and Inverse Limits (Category Version)

Category: $\underset{\text{objects}}{\mathcal{O}}$ $\underset{\text{morphisms}}{\mathcal{M}}$

Given $\alpha_k : O_k \to O_{k+1}$, with objects $O_k \in \mathcal{O}$, and morphisms $\alpha_k \in \mathcal{M}$ for $k \in \mathbb{N}$, the *direct limit* $\lim\limits_{\rightarrow} (O_k, \alpha_k) = \big(O, (m^k)_k\big)$, where $O \in \mathcal{O}$ and the $m^k \in \mathcal{M}$, $m^k : O_k \to O$, are such that $m^{k+1} \circ \alpha_k = m^k$; and whenever $\big(B, (b^k)_k\big)$, $B \in \mathcal{O}$ and $b^k \in \mathcal{M}$ also satisfy $b^{k+1} \circ \alpha_k = b^k$, there is a unique morphism $f : O \to B$ such that $f \circ m^k = b^k$, $k \in \mathbb{N}$.

$$\begin{tikzcd}
O_k \arrow[rr, "\alpha_k"] \arrow[dr, "m^k"] \arrow[ddr, "b^k"'] & & O_{k+1} \arrow[dl, "m^{k+1}"'] \arrow[ddl, "b^{k+1}"] \\
& O \arrow[d, "f"] & \\
& B &
\end{tikzcd}$$

Direct limit diagram (commuting)

For the definition of an *inverse limit* just reverse all the arrows (exercise).

(Returning to the proof) $f : \mathbb{R}^n \to Y$ is k-$\mathcal{P}$-e, $k \in \mathbb{N}$, if $f(x)$ depends only on $t_k(\mathcal{P} - x)$; $t_k(Q) = k$-decorated tile, where the tile is the Voronoi domain of Q containing 0.

$$C_{k\text{-}\mathcal{P}\text{-e}}\left(\mathbb{R}^n, Y\right) = C\left(\Gamma_k, Y\right)$$

so

$$C_{\text{s-}\mathcal{P}\text{-e}}\left(\mathbb{R}^n, Y\right) = \bigcup_k C_{k\text{-}\mathcal{P}\text{-e}}\left(\mathbb{R}^n, Y\right) = \lim_{\leftarrow}\left(C\left(\Gamma_k, Y\right), \alpha_k^*\right)$$

$$\Gamma_{k+1} \xrightarrow{\alpha_k} \Gamma_k \;\rightsquigarrow\; C\left(\Gamma_k, Y\right) \xrightarrow{\alpha_k^*} C\left(\Gamma_{k+1}, Y\right), \quad \alpha_k^*(f) = f \circ \alpha_k$$

Now a quick proof that the spectrum is $\Omega_{\mathcal{P}}$.

Reminder of the notion of spectrum. If $\mathcal{A}$ is a commutative C*-algebra with 1, $\operatorname{Spec}\mathcal{A} = \{\underbrace{\text{non-zero characters}}_{*\text{-algebra morphisms } \mathcal{A}\to\mathbb{C}}\} \subset \mathcal{A}'$ (dual of $\mathcal{A}$ with the weak-$*$-topology).

The dual of a normed space is the same as the dual of its completion.

1) If $\mathcal{B}$ is a dense subalgebra of $\mathcal{A}$ then $\operatorname{Spec}(\mathcal{B}) = \operatorname{Spec}(\mathcal{A})$
2) If $\mathcal{B} \xrightarrow{\phi} \mathcal{A}$, then $\phi^* : \mathcal{A}' \to \mathcal{B}'$ and $\phi^*|_{\operatorname{Spec}(\mathcal{A})} : \operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{B})$ is continuous in the weak-$*$-topology (because it is continuous in the norm topology on $\mathcal{B}'$). $\operatorname{Spec}(\mathcal{A})$ has the relative weak-$*$-topology.
3) If $\mathcal{A}_k \xrightarrow{\phi_k} \mathcal{A}_{k+1}$, then
$$\operatorname{Spec}\left(\lim_{\rightarrow}(\mathcal{A}_k, \phi_k)\right) = \lim_{\leftarrow}\left(\operatorname{Spec}(\mathcal{A}_k), \phi_k^*|_{\operatorname{Spec}(\mathcal{A}_k)}\right)$$

Hence to calculate $\operatorname{Spec}\left(C_{\mathcal{P}}(\mathbb{R}^n)\right)$:

$$\begin{aligned}
\operatorname{Spec}\left(C_{\text{s-}\mathcal{P}}(\mathbb{R}^n)\right) &\underset{\text{see above}}{=} \operatorname{Spec}\left(\lim_{\rightarrow}\left(C(\Gamma_k), \alpha_k^*\right)\right) \\
&= \lim_{\leftarrow}\left(\operatorname{Spec}(C(\Gamma_k)), (\alpha_k^*)^*|_{\operatorname{Spec}\ldots}\right) \\
&\underset{\text{see below}}{=} \lim_{\leftarrow}(\Gamma_k, \alpha_k)
\end{aligned}$$

Regarding the last equality.

$$\begin{aligned}
X \text{ compact Hausdorff} &\implies X \cong \operatorname{Spec}(C(X)),\ x \mapsto \mathrm{ev}_x \\
&\implies \Gamma_k \cong \operatorname{Spec}(C(\Gamma_k))
\end{aligned}$$

So

$$\begin{aligned}
\eta \in \operatorname{Spec}(C(\Gamma_{k+1})) &\Rightarrow \eta = \mathrm{ev}_x \text{ for some } x \in \Gamma_{k+1} \\
&\Rightarrow (\alpha_k^*)^*|_{\operatorname{Spec}}(\eta) = (\alpha_k^*)^*|_{\operatorname{Spec}}(\mathrm{ev}_x) \\
&= \mathrm{ev}_x \circ \alpha_k^*
\end{aligned}$$

$$\begin{array}{ccc} \Gamma_{k+1} & \xrightarrow{\alpha_k} & \Gamma_k \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Spec}(C(\Gamma_{k+1})) & \xrightarrow{(\alpha_k^*)^*|_{\mathrm{Spec}}} & \mathrm{Spec}(C(\Gamma_k)) \end{array}$$

$$\varprojlim (\mathrm{Spec}(C(\Gamma_k)), (\alpha_k^*)^*|_{\mathrm{Spec}\dots}) \cong \varprojlim(\Gamma_k, \alpha_k)$$

and

$$\begin{aligned} f \in C(\Gamma_k) &\Rightarrow (\mathrm{ev}_x \circ \alpha_k^*)(f) = \mathrm{ev}_x(f \circ \alpha_k) = f(\alpha_k(x)) = \mathrm{ev}_{\alpha_k(x)}(f) \\ &\Rightarrow (\alpha_k^*)^*|_{\mathrm{Spec}} \cong \alpha_k \end{aligned}$$

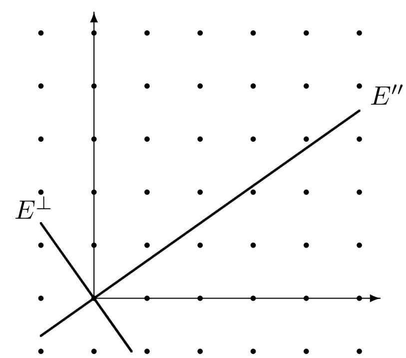
9 (9 February 2009)

Cut + Project Tilings

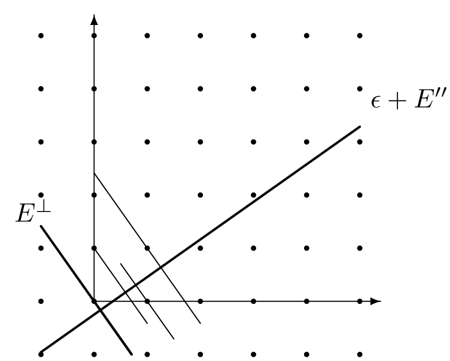
Canonical cut + project tiling.

$$\underset{=E^\perp}{\mathbb{R}^n} \xleftarrow{\text{proj.}} \underset{\supset\mathbb{Z}^N \text{ lattice}}{\mathbb{R}^N} \xrightarrow{\text{proj.}} \mathbb{R}^d =: E'' \quad (d < N)$$

$\tilde{\mathcal{T}}$ is a $\mathbb{Z}^N$-periodic tiling of $\mathbb{R}^N$ such that there are finitely many tiles up to translation, each of which is of the form $\mathsf{k} = \underset{\text{poly'n in } E^\perp}{\mathsf{k}^\perp} \times \underset{\text{poly'n in } E''}{\mathsf{k}''}$. ("$\mathsf{k}$" for the German "klotz")



Cut and Project: $N = 2$, $d = 1$



Cut and Project with $(\epsilon + E'') \cap \mathbb{Z}^2 = \emptyset$

Choose $\epsilon \in \mathbb{R}^N$: look at intersection between

$$(\epsilon + E'') \cap \tilde{\mathcal{T}}$$

Assumption $(\epsilon + E'') \cap \mathbb{Z}^2 = \emptyset$

if $\epsilon + E''$ cuts through $\mathbb{Z}^2$ then $\epsilon + E''$ cuts through the boundary of two neighboring tiles; want to avoid this so that's why we put the ϵ in.

Try to understand the inverse limit space.

$$f \in C_{0\text{-}\mathcal{P}\text{-e}}(\overset{E''}{\mathbb{R}}) \text{ defines a function } \tilde{f} \text{ on a dense set of } \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$$

$$C_{0\text{-}\mathcal{P}\text{-e}}(\mathbb{R}) \overset{i}{\subset} C(\underset{\text{0-cut}}{\mathbb{T}^2_{\text{0-c}}})$$

$$\underbrace{\text{Spec}\left(C(\mathbb{T}^2_{\text{0-c}})\right)}_{\cong \mathbb{T}^2_{\text{0-c}}} \xrightarrow{i^*|_{\text{Spec}}} \text{Spec}\left(C_{0\text{-}\mathcal{P}\text{-e}}(\mathbb{R})\right), \quad x \in \mathbb{T}^2_{\text{0-c}} \mapsto \text{ev}_x$$

$$i^*\text{ev}_x = \text{ev}_x \circ i$$

$$i^*\text{ev}_x = i^*\text{ev}_y \text{ if } x \underset{\text{blue line}}{\sim} y \text{ or } x \underset{\text{green line}}{\sim} y$$

Delone set of $\mathcal{P}$ given by the vertices is

$$(\epsilon + E'') \cap (\mathbb{Z}^2 + \bigcup_{\mathsf{k} \text{ tile in } \tilde{\mathcal{T}}} \mathsf{k}^\perp)$$

Canonical choice: $\pi^\perp$(unit cube)

$$\pi^\perp : \mathbb{R}^N \to E^\perp \text{projection along } E''$$

The thing I think should be canonical

monical: $\mathsf{k}^\perp$ polyhedra s.th. $\partial \mathsf{k}^\perp = \bigcup_{\text{faces}} f_i$

Look at the stabilizer

$$\text{Stab}_{\pi^\perp(\mathbb{Z}^N)}(\text{affine space spanned by all } f_i > 0)$$

The rank of the stabilizer determines the complexity of the tiling.

Back to the construction of the algebra.

$$\mathcal{A}_\mathcal{P} = \underset{(\alpha_\xi(f))(x) = f(x-\xi)}{C_\mathcal{P}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n} \cong \underset{\phi \text{ induced by } \omega \in \Omega_\mathcal{P} \mapsto \omega - x}{C(\Omega_\mathcal{P}) \rtimes_\phi \mathbb{R}^n}$$

Fix isomorphism $C(\Omega_\mathcal{P}) \xrightarrow{\sigma} C_\mathcal{P}(\mathbb{R}^n)$, a $*$-isomorphism of C*-algebras: $\sigma(\tilde{f})(x) := \tilde{f}(\mathcal{P} - x)$

We can see why $C_\mathcal{P}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n$ is a good algebra.

Example.

$$\mathcal{P} = \{x\}\,,\ C_{\{x\}}(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)^{+^{\text{(adjoin 1)}}} = C_0(\mathbb{R}^n) + \underset{\text{multiples of 1}}{\mathbb{C}}$$

$$\mathcal{A}_{\{x\}} = \underset{\text{quant'n of a classical particle}}{C_0(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n} + \mathbb{C} \rtimes \mathbb{R}^n$$

Recall. We took the representation induced by $\mathrm{ev}_0 : C_{\{x\}}(\mathbb{R}^n) \to \mathbb{C}$. This gave the Schrödinger representation for the first part of $\mathcal{A}_{\{x\}}$.

$$\underset{\text{character rep'n}}{C_{\mathcal{P}}(\mathbb{R}^n)} \rtimes \underset{\text{induced rep'n}}{\mathbb{R}^n}$$

The representations on $\mathcal{A}_{\mathcal{P}}$ we look at are induced representations of the character irreducible representations on $C_{\mathcal{P}}$.

So $\forall \omega \in \Omega_{\mathcal{P}} = \mathrm{Spec}\,(C_{\mathcal{P}}(\mathbb{R}^n))$ we have an irreducible representation $\mathrm{ev}_\omega \circ \sigma^{-1}$ and therefore an irreducible representation π_ω of $\mathcal{A}_{\mathcal{P}}$.

Let $F \in \mathcal{A}_{\mathcal{P}}$ of the form $F : \mathbb{R}^n \to C_{\mathcal{P}}(\mathbb{R}^n)$

$$\begin{aligned} \tilde{F}(\xi) &= \sigma^{-1} \circ F(\xi) \\ \tilde{F} &: \mathbb{R}^n \to C(\Omega_{\mathcal{P}}) \\ \pi_\omega &: \mathcal{A}_{\mathcal{P}} \to B(L^2(\mathbb{R}^n)) \\ (\pi_\omega(\tilde{F})\psi)(x) &= \int \mathrm{d}y\ \underbrace{\tilde{F}(x-y)(\omega+x)}_{\text{integral kernel of } \pi_\omega(\tilde{F})}\ \psi(y) \end{aligned}$$

First important consequence of this:

Consider $\omega = \mathcal{P}$ and

$$H = \underset{\text{in the Sobolev rep'n}}{\frac{\hat{p}^2}{2m} + V(\hat{q})} \quad , V \in C_{\mathcal{P}}(\mathbb{R}^n, \mathbb{R})$$

Then $\forall g \in C_0(\mathbb{R})$,

$$\underset{\text{spec calc}}{g(H)} \in \pi_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}})$$

Use Laplace transform.

10 (11 February 2009)

The construction of the algebra $\mathcal{A}_\mathcal{P}$ (cont.)

We have this algebra

$$\begin{aligned} \mathcal{A}_\mathcal{P} &= C_\mathcal{P}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n \cong C(\Omega_\mathcal{P}) \rtimes_\alpha \mathbb{R}^n \\ \pi_\mathcal{P} &: \mathcal{A}_\mathcal{P} \to B(L^2(\mathbb{R}^n)) \\ f &: \mathbb{R}^n \to C_\mathcal{P}(\mathbb{R}^n) \\ (\pi_\mathcal{P}(f)\psi)(x) &= \int f(x-y)(x)\psi(y)\mathrm{d}y \end{aligned}$$

Theorem. *If $H = \frac{\hat{p}^2}{2m} + V(\hat{q})$, $V \in C_\mathcal{P}(\mathbb{R}^n)$, then $\forall F \in C_0(\mathbb{R})$, $F(H) \in \pi_\mathcal{P}(\mathcal{A}_\mathcal{P})$ ($\Leftrightarrow \exists A \in \mathcal{A}_\mathcal{P}$ such that A is represented by $F(H)$).*

Proof. $\hat{p} = \frac{\hbar}{i}\Delta$, $\hat{q}$ left multiplication by $x \mapsto x$. Taking $\hbar = 1$, $m = 1$, $H = -\Delta + V$, then H is bounded below (but not above), and by a shift of V, $H \geq 0$.

Suppose F has a Laplace transform; F is regular enough that

$$\tilde{F}(t) = 2\pi \int_{-\infty}^{\infty} \mathrm{e}^{tE} F(E)\mathrm{d}E$$

$$\left(F(E) = \int_0^\infty \mathrm{e}^{-tE}\tilde{F}(t)\mathrm{d}t \text{ is the Laplace transform of } \tilde{F} : \mathbb{R} \to \mathbb{R} \right)$$

Then $F(H)$ is defined by "Laplace" functional calculus

$$F(H) = \int_0^\infty \mathrm{e}^{-tH}\tilde{F}(t)\mathrm{d}t; \text{ bounded because } H \text{ is bounded below.}$$

To show that F belongs to a norm-closed subalgebra it suffices to show that e^{-tH} belongs; that is, for these functions F, it suffices to show that $\mathrm{e}^{-tH} \in \pi_\mathcal{P}(\mathcal{A}_\mathcal{P})$.

Consider first the case where

$$\mathrm{e}^{-t(-\Delta)} \text{ (the heat kernel)}^{1}$$

Lemma.

$$\left(\mathrm{e}^{t\Delta\psi}\right)(x) = \int \underbrace{\left(\mathrm{e}^{t\Delta}\right)_{xy}}_{=\frac{c_1}{t^{n/2}}\mathrm{e}^{\frac{-(x-y)^2}{4t}}} \psi(y)\mathrm{d}y$$

---

[1] With $\hbar = 1$, $\hat{p}^2 = -\Delta$

$$\underset{f(x-y)(x)}{\pi_{\mathcal{P}}(f)} = \mathrm{e}^{t\Delta} \quad \underset{f(\xi)(x)=\frac{c_1}{t^{n/2}}\mathrm{e}^{\frac{-\xi^2}{4t}}}{f:\ \mathbb{R}^n \to C_{\mathcal{P}}(\mathbb{R}^n)}$$

If we have a constant potential (zero potential)

$$\mathrm{e}^{-tH} \in \underset{\substack{\cong C_0(\hat{\mathbb{R}}^n) \\ = \text{ functions of } \hat{p}}}{\mathbb{C} \rtimes \mathbb{R}^n} \subset \mathcal{A}_{\{x\}}$$

So [Dyson-Phillips expansion]

$$\mathrm{e}^{-tH} = \mathrm{e}^{-t\Delta - tV} = \mathrm{e}^{t\Delta} + \sum_{\nu=0}^{\infty} \int \mathrm{d}s_0 \cdots \underset{s_i \geq 0\ \sum_0^\nu s_i = t}{\int \mathrm{d}s_\nu} \mathrm{e}^{s_1\Delta} V \mathrm{e}^{s_2\Delta} V \cdots V \mathrm{e}^{s_\nu\Delta}$$

If V is bounded, the series converges in norm. $-\Delta \geq 0 \Rightarrow \|\mathrm{e}^{s\Delta}\| \leq 1$

$$\| \int \mathrm{d}s_0 \cdots \int \mathrm{d}s_\nu \mathrm{e}^{s_1\Delta} \cdots \| \leq \underset{\substack{s_i \geq 0\ \sum_0^\nu s_i = t \\ \text{volume of a } \nu\text{-simplex}}}{\int \mathrm{d}s_1 \cdots \int \mathrm{d}s_\nu} \ \|V\|^\nu \leq \frac{t^\nu}{\nu!}\|V\|^\nu$$

So the series converges absolutely like $\sum \frac{t^\nu}{\nu!}\|V\|^\nu \leq \mathrm{e}^{t\|V\|}$

Now we are almost done. Need to show $\mathrm{e}^{s_i\Delta}$ is in the algebra, so that $\mathrm{e}^{s_1\Delta}V\mathrm{e}^{s_2\Delta}V \cdots \mathrm{e}^{s_\nu\Delta}$ is in the algebra.

It suffices to show that $V\mathrm{e}^{t\Delta} \in \pi_{\mathcal{P}}(\mathcal{A}_{\mathcal{P}})$:

$$(V\mathrm{e}^{t\Delta})\psi(x) = \int \underbrace{V(x)\frac{c_1}{t^{n/2}}\mathrm{e}^{\frac{-|x-y|^2}{4t}}}_{\text{integral kernel } f(x-y)(x)} \psi(y)\mathrm{d}y$$

$$f : \mathbb{R}^n \to C_{\mathcal{P}}(\mathbb{R}^n)$$

$$f(\xi)(x) = V(x)\frac{c_1}{t^{n/2}}\mathrm{e}^{\frac{-\xi^2}{4t}}$$

$$\text{then, } \pi_{\mathcal{P}}(f) = V\mathrm{e}^{t\Delta}$$

QED $\square$

Q: This is a good representation (Schrödinger); what about the others? (Other representations from other characters.)

Covariant families of operators

Comes from the physics of condensed matter. Random potentials are described by potentials which are random variables over some probability space $(\Omega, \mathbb{P})$. So

$$\Omega \ni \omega \mapsto V_\omega : \mathbb{R}^n \to \mathbb{R}$$

Ideally, solve everything for $H_\omega = H_0 + V_\omega$ ($H_0 = -\Delta$ for example).

$$\rightsquigarrow \int_\Omega \mathrm{Tr}(\rho_\omega A_\omega) \mathrm{d}\mathbb{P}$$

The density matrix might depend on ω.

Homogeneous media = microscopically translation invariant.

Idea: $(\Omega, \mathbb{P})$ should carry an action of $\mathbb{R}^n$: $\omega \overset{\xi}{\mapsto} \omega - \xi$ such that

$$\forall \omega \in \Omega \ x \in \mathbb{R}^n : V_{\omega - x} = U_x V_\omega U_{-x} \,, \ ((U_x \psi)(y) = \psi(y - x))$$

Covariant system. Bellisard, Johnson and Moser: for aperiodic media take $\Omega =$ hull. $\mathbb{P}$ should be translation invariant. Pure phases = uniquely ergodic $\mathbb{P}$ with respect to itself.

11 (Friday, 13 February 2009) (Thanks to Carl Olimb for the notes.)

Message (from last time): For aperiodic solids (which are homogeneous) the quantum mechanical system is described by a covariant family of operators $H = (H_\omega)_{\omega\in\Omega}$, $(\Omega, \mathbb{P}, \alpha, \mathbb{R}^n)$; $\mathbb{P}$ is a probability measure. For us, $\Omega = \Omega_{\mathcal{P}}$. $H_{\omega-x} = U_x H_\omega U_x^{-1}$. (The idea is, this $\mathbb{P}$ is motivated by physics.)

Recall $\forall\omega \in \Omega$, there was the representation

$$\pi_\omega : C(\Omega_{\mathcal{P}}) \rtimes_\alpha \mathbb{R}^n \to \mathcal{B}(L^2(\mathbb{R}^n))$$

induced by the $\mathrm{ev}_\omega : C(\Omega_{\mathcal{P}}) \to \mathbb{C}$. For f sufficiently regular,

$$(\pi_\omega(f)\psi)(x) = \int \mathrm{d}y \underbrace{f(x-y)(\omega+x)}_{\pi_\omega(f)_{xy}}\psi(y)$$

If we shift ω the representation is $\mathcal{P}$-equivariant: $\pi_{\omega-x}(f) = U_x\pi_\omega(f)U_{-x}$, where U_x is the representation (exercise).

Corollary. *$\forall A \in \mathcal{A}$, $(\pi_\omega(A))_\omega$ is a covariant family of operators on $L^2(\mathbb{R}^n)$.*

Corollary. $\left(\det\left(H_\omega = \dfrac{\hat{p}^2}{2m} + V_\omega\right)\right)_\omega$ *is a covariant family of Hamiltonians. Then $\forall F \in C_0(\mathbb{R})\exists h : \mathrm{e}^{-tH\omega} = \pi_\omega(\mathrm{e}^{-th})$. ($\mathrm{e}^{-th} \in \mathcal{A}_{\mathcal{P}}$)*

We have to be careful about the isomorphism $C(\Omega_{\mathcal{P}}) \xrightarrow{\sigma} C_{\mathcal{P}}(\mathbb{R}^n)$, $\sigma(f)(x) = f(\mathcal{P} - x)$. If V is $\mathcal{P}$-equivariant then $U = \sigma^{-1}(V)$. Then $h = \dfrac{\hat{p}^2}{2m} + V$.

Back to $\mathbb{P}$. "measurement is related to traces"

Traces.

Definition. A *trace* on a (separable) Hilbert space $\mathcal{H}$ is a positive linear functional $\mathrm{Tr} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ such that $\forall U \in \mathcal{U}(\mathcal{H})$ and $\forall A$ such that A is *traceclass*, $\mathrm{Tr}(UAU^{-1}) = \mathrm{Tr}(A)$.

A is traceclass if $|A| = \sqrt{A^*A}$ has $\mathrm{Tr}(|A|) < \infty$.

Q. How many traces are there on $\mathcal{H}$?

1. Operator trace:

$$\underbrace{\mathcal{K}(\mathcal{H})}_{\textit{compact}\text{ operators}} \subset \mathcal{B}(\mathcal{H})$$

Think of $\mathcal{K}(\mathcal{H})$ as the norm closure of operators with finite dimensional image.

For a positive operator $A \in \mathcal{K}(\mathcal{H})$ look at its spectrum $\sigma(A) = \{ \lambda_n \mid n \in \mathbb{N}\}$; $\lambda_n \in \mathbb{R}$ and 0 is an accumulation point.

Let $(\mu_n)_n$ be a decreasing sequence exhausting $\sigma(A)$ and taking care of multiplicity. Then define

$$\mathrm{Tr}(A) = \sum \mu_n \,, (\text{ possibly finite or infinite})$$

If finite, then we have traceclass A.

$$A \text{ traceclass } \Leftrightarrow (\mu_n)_n \in \ell^1 \qquad \text{“you can sum it”}$$

Extend trace by linearity. Then Tr is a linear functional with dense domain called $\mathcal{L}^1(\mathcal{H})$

$$\mathcal{L}^1(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H}) \mid |A| \text{ traceclass}\}$$

Define $\mathcal{L}^p(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid |A|^p \text{ is traceclass}\}$. These operators are all compact. Notice that

$$\mathcal{L}^p \underset{\text{ideal}}{\subset} \mathcal{L}^{p'} \underset{\text{ideal}}{\subset} \mathcal{K}(\mathcal{H}) \underset{\text{ideal}}{\subset} \mathcal{B}(\mathcal{H}) \;\text{ if } p \geq p' \geq 1$$

Tr is not continuous in norm. We could put other norms on these. $\|A\|_p = \mathrm{Tr}(|A|^p)$ is a norm on $\mathcal{L}^p(\mathcal{H})$ making it into a Banach space.

Consider $\mathcal{L}^2(\underbrace{L^2(X, \underset{\text{measure}}{\nu})}_{\text{Hilbert space}})$. Think $X = \mathbb{R}^n$ and $\nu =$ Lebesgue.

If $A \in \mathcal{B}(L^2(X,\nu))$ has an integral kernel A_{xy}, then

$$A \in \mathcal{L}^2(L^2(X,\nu)) \quad \Longleftrightarrow \quad (x,y) \mapsto A_{xy} \;\in L^2(X \times X, \nu \times \nu)$$

$(\forall\, \psi : (A\psi)(x) = \int A_{xy}\psi(y)\ )$

Then $\mathrm{Tr}(A) = \int \mathrm{d}\nu(x) A_{xx}$ doesn't work in general. But

Theorem. *If in addition* $(x,y) \to A_{xy}$ *is continuous then* $\mathrm{Tr}(A) = \int \mathrm{d}\nu(x) A_{xx}$.

Besides Tr there is $c\mathrm{Tr}$, $\forall\, c > 0$ (a class of traces).

2. <u>Dixmier trace</u>: is a trace that vanishes on $\mathcal{L}^1(\mathcal{H})$. (This will be talked about later.)

Now, traces on C*-algebras.

Definition. A *trace* T on a C*-algebra $\mathcal{A}$ is a positive linear functional such that $\forall\, A \in \mathcal{A}$ traceclass $(T(|A|) < \infty)$ and $B \in \mathcal{A}$, $T(AB) = T(BA)$.

If $\mathcal{A} = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \subset M_5(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ we can take $c_1\mathrm{Tr}_{\mathbb{C}^2} + c_2\mathrm{Tr}_{\mathbb{C}^3}$ as a trace $\forall\, c_1, c_2 > 0$.

Notice again, these functionals are not bounded and this T need not be continuous.

Example. $\mathcal{A} = C(X)$, X compact Hausdorff. Bounded traces are therefore positive elements of $\mathcal{A}'$.
$\Longleftrightarrow$
positive measures on X.
Why? If ν is such a measure then define $T(f) = \int_X f(x) \mathrm{d}\nu(x)$ $(f \in C(X))$.

Example. Take $(\mathcal{B}, \alpha, \mathbb{Z}^n)$ (C*-algebra with $\mathbb{Z}^n$-action), suppose T trace on $\mathcal{B}$ which is invariant: $T = T \circ \alpha$. Then define $\tilde{T} : \mathcal{B} \rtimes_\alpha \mathbb{Z} \to \mathbb{C}$, where for $f : \mathbb{Z} \to \mathcal{B}$, $\tilde{T}(f) = T(f(0))$.

Example. Same thing for $(\mathcal{B}, \alpha, \mathbb{R}^n)$ leads to unbounded traces, always.

12 (18 February 2009)

Trace-class, cont.

Suppose we have $(\mathbb{P}, \Omega, \alpha, \mathbb{R}^n)$, $\mathbb{P}$ an invariant probability measure on Ω (σ-algebra of Borel sets) gives us a trace $\rightsquigarrow \tau$ trace on $C(\Omega) \rtimes_\alpha \mathbb{R}^n$:

$$\text{if } f : \mathbb{R}^n \to C(\Omega) \text{ is continuous, then}$$
$$\tau(f) = \int_\Omega f(0)(\omega)\mathrm{d}\mathbb{P}(\omega)$$

This is a special case of a noncommutative dynamical system (ncd)

$$(\underbrace{\mathcal{B}}_{\text{C*-algebra}}, \alpha, \mathbb{R}^n)$$

$$\text{and a trace } \tau : \mathcal{B} \to \mathbb{C} \text{ which is } \alpha \text{ invariant: } \tau \circ \alpha = \tau$$
$$\text{and then } \tilde{\tau} : \mathcal{B} \rtimes_\alpha \mathbb{R}^n \to \mathbb{C}\,,\ \tilde{\tau}(f) = \tau(f(0))$$
$$\forall f : \mathbb{R}^n \to \mathcal{B} \text{ such that } f(0) \text{ is } \tau \text{ trace-class.}$$

Trace per unit volume.

The idea behind this is that in physics there are *extensive* and *intensive* quantities.

$$\text{extensive quantity} = \underset{N\ \#\text{ of particles, } V \text{ volume, } E}{\text{scales linearly with system size}}$$
$$\text{intensive quantity} = \underset{\text{often an ext. quant. normalized: } N/V \text{ density}}{\text{doesn't depend on system size}}$$

Let $\chi : \mathbb{R}^n \to \mathbb{R}$, $\chi \in L^1$, $\int \chi = 1$ (e.g., $\chi = \mathbf{1}_{[0,1]^n}$).

Expectation looks something like

$$\underbrace{\int_\Omega \mathrm{d}\mathbb{P}(\omega)\mathrm{Tr}(A_\omega \chi)}_{\text{localize } A_\omega \text{ with } \chi} = \int \mathrm{d}\mathbb{P}(\omega) \int \mathrm{d}x A_{\omega,xx}\chi(x)$$
$$\overset{\text{Fubini}}{=} \int \mathrm{d}x \left(\underbrace{\int \mathrm{d}\mathbb{P}(\omega) A_{\omega,xx}} \right) \chi(x)$$
$$= \int \mathrm{d}x \left(\int \mathrm{d}\mathbb{P}(\omega) A_{\omega-x,00} \right) \chi(x)$$
$$= \int \mathrm{d}x \left(\int \mathrm{d}\mathbb{P}(\omega - x) A_{\omega-x,00} \right) \chi(x)$$
$$= \int \mathrm{d}x \left(\int \mathrm{d}\mathbb{P}(\omega) A_{\omega,00} \right) \chi(x)$$

Recall that $\int \mathrm{d}\mathbb{P}(\omega) A_{\omega,00} = \tau(A)$, where $A \in \mathcal{A}_{\mathcal{P}}$ such that $\pi_\omega(A) = A_\omega$.

Measure quantity on $[0,1]^n$; but doesn't depend on $\mathbf{1}_{[0,1]^n}$ because of homogeneity.

$(\Lambda_n)_n$ sequence of cubes such that $\bigcup_n \Lambda_n = \mathbb{R}^n$ (Foellner sequence; van Hove sequence). Try to take

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Lambda_n)} \mathrm{Tr}(A\mathbf{1}_{\Lambda_n}) \quad (A \text{ operator on } \mathcal{B}(L^2(\mathbb{R}^n)))$$

trace/volume of A

Now suppose that $\mathbb{P}$ is ergodic. Apply Birkhoff to

$$\begin{aligned}
\int \mathrm{d}\mathbb{P}(\omega) A_{\omega,00} &= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Lambda_n)} \int_{\Lambda_n} \mathrm{d}x A_{\omega-x,00} \\
&= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Lambda_n)} \underbrace{\int_{\Lambda_n} \mathrm{d}x A_{\omega,xx}}_{\mathrm{Tr}(A_\omega \mathbf{1}_{\Lambda_n})} \\
&= \frac{\text{trace}}{\text{volume}}(A)
\end{aligned}$$

Suppose that the observable is a function of the Hamiltonian:

$$\underset{\text{observable}}{A} = f(\underset{\text{Hamiltonian}}{H})$$

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(\Lambda_n)} \left(\mathrm{Tr}(f(H)\mathbf{1}_{\Lambda_n}) - \mathrm{Tr}(f(H|_{\Lambda_n}))\right) \overset{n\to\infty}{\longrightarrow} 0 \\
&H|_{\Lambda_n} = \mathbf{1}_{\Lambda_n} H \mathbf{1}_{\Lambda_n} \\
&\mathbf{1}_{\Lambda_n} : L^2(\mathbb{R}^n) \overset{\text{projection}}{\rightarrow} L^2(\Lambda_n)
\end{aligned}$$

Could try looking at trace per unit volume of H itself:

$$\frac{1}{\mathrm{vol}(\Lambda_n)} \mathrm{Tr}\left(H|_{\Lambda_n}\right)$$

which is a spectral measure for H; allows us to calculate (by integration over spec of H) all trace/vol of measurable functions of H.

Integrated density of states (IDS)

$$\text{density of states at } E \pm \Delta = \frac{\text{no. of states of } H \text{ of energy in } E + \Delta}{\text{volume} \cdot |\Delta|}$$

terribly undefined: e.g. non-discrete spectrum, ...

$$\mathrm{DS}(E) = \text{density of states at } E \pm \Delta$$

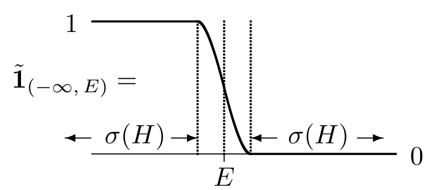
$$\mathrm{IDS}(E) = \int_{-\infty}^{E} \mathrm{DS}(E')\mathrm{d}E'$$

$$:= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Lambda_n)} (\#\text{ eigenstates of } H|_{\Lambda_n} \text{ of energy } \leq E)$$

$$\big(\mathbf{1}_{[-\infty,E]}(H|_{\Lambda_n}) = \text{spectral projection of } H|_{\Lambda_n} \text{ to energy } \leq E\big)$$

$$\overset{\text{Shubin's formula}}{=} \lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Lambda_n)} \mathrm{Tr}\left(\mathbf{1}_{[-\infty,E]}\left(H|_{\Lambda_n}\right)\right)$$

Suppose that $E \notin \underbrace{\sigma(H)}_{\text{spec of } H}$ (allows replacing $\mathbf{1}$ with $\tilde{\mathbf{1}}$, below) and $H = (H_\omega)_\omega$ are covariant operators bounded from below.



Smooth version of $\mathbf{1}_{(-\infty,\,E)}$

Then $\exists h \in \mathcal{A}_{\mathcal{P}}$ such that $\tilde{\mathbf{1}}_{(-\infty,E)}(H_\omega) = \pi_\omega(h)$ and

Theorem. $\mathrm{IDS}(E) = \tau(h)$

h is a projection in $\mathcal{A}_{\mathcal{P}}$ as the direct sum of π_ω is faithful and $\tilde{\mathbf{1}}_{(-\infty,\,E)}(H)$ is a spectral projection.

13 (20 February 2009)

Quasicrystals (QC)

Approximate the infinite QC by something finite using some sort of boundary conditions.

IDS(E) integrated density of states up to energy E
DS(E) density of states

$$\mathrm{QC} : \mathcal{P} = \begin{cases} \text{octonacci chain} & \text{1-d} \quad \text{(looks like the devil's staircase)} \\ \text{octagonal tiling} & \text{2-d} \\ \text{icosahedral tiling} & \text{3-d} \end{cases}$$

$$H = \frac{\hat{p}^2}{2m} + V\,, \;\; V \; \mathcal{P}\text{-equivariant}$$

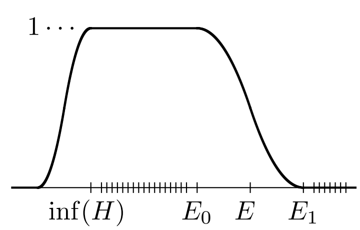
Discretizing H; becomes bounded; just a big matrix; diagonalize.

K_0-theoretic Gap-labelling

H bounded from below (Hamiltonian) is *affiliated with* a C*-algebra $\mathcal{A}$ if $F(H) \in \mathcal{A}\ \forall\ F \in C_0(\mathbb{R})$ (if H is bounded then this means that $H \in \mathcal{A}$).

Suppose that $E \notin \sigma(H)$. Then there is a function $F \in C_0(\mathbb{R})$ such that

$$F(x) = \begin{cases} 1 & \text{if } x \in \sigma(H) \cap (-\infty, E] \\ 0 & \text{if } x \in \sigma(H) \cap (E, \infty) \end{cases}$$



Hence $F(H) \in \mathcal{A}$ (F is continuous).

Remark.

i) $F(H)F(H) = F(H)^2 = F(H)$ since $F(H) = 0$ or 1 on $\sigma(H)$; $F(H)^* = F(H)$, so $F(H)$ is a projection.

ii) If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation, then $\pi(F(H))$ is the spectral projection of $\pi(H)$ onto the eigenspace of energies $\leq E$. Denote it by $P_{\leq E}(H)$.

iii) $P_{\leq E}(H)$ exists for any H (by functional calculus with measurable functions)

But in general $P_{\leq E}(H) \notin \pi(\mathcal{A})$ if $E \in \sigma(H)$. However always

$$P_{\leq E}(H) \in \overline{\pi(\mathcal{A})}^{\text{SOT}} \quad \text{(a von Neumann algebra)}$$

Want to measure the difference between $E \in \sigma(H)$ and E in a gap.

Recall that the elements of $K_0(\mathcal{A})$ are formal differences of equivalence classes of projections. $K_0(\mathcal{A})$ carries an "order" structure; there is a positive cone $K_0(\mathcal{A})^+$ such that

i) $K_0(\mathcal{A})^+ + K_0(\mathcal{A})^+ \subset K_0(\mathcal{A})^+$

ii) $K_0(\mathcal{A})^+ \cap (-K_0(\mathcal{A})^+) = \{0\}$

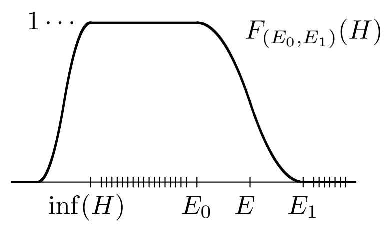
iii) $K_0(\mathcal{A})^+ + (-K_0(\mathcal{A})^+) = K_0(\mathcal{A})$

In general an element of $K_0(\mathcal{A})$ is of the form $[p]_0 - [q]_0$. The elements of the cone $K_0(\mathcal{A})^+$ are of the form $[p]_0 - [0]_0 = [p]_0$.

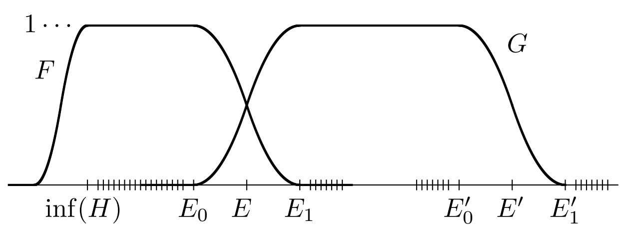
Abstract K_0-theoretic gap-labelling

The gaps of H are labelled by an order preserving map

$$\mathcal{L} : \underbrace{\mathrm{gap}(H)}_{\substack{= \text{ gaps of } \sigma(H); \\ \text{open intervals ordered by the order on } \mathbb{R}}} \longrightarrow (K_0(\mathcal{A}), K_0(\mathcal{A})^+);\ (E_0, E_1) \overset{\mathcal{L}}{\mapsto} [F_{(E_0,E_1)}(H)]_0$$



Order preserving is almost obvious. Gaps $(E_0, E_1) < (E_0', E_1')$ lead to $F' = F + G$ with F and G as shown:



$$F'(H) = F(H) + G(H) \quad \Longrightarrow \quad [F'(H)]_0 = [F(H)]_0 + [G(H)]_0$$

because $F(H)$ and G(H) are orthogonal projections. So,

$$[F'(H)]_0 - [F(H)]_0 \in K_0(\mathcal{A})^+ \quad \Longleftrightarrow \quad [F'(H)]_0 > [F(H)]_0$$

An important remark. A gap label $\mathcal{L}((E_0, E_1))$ is a topological invariant in the following sense. Suppose we perturb H by $(H_t)_{t\in[0,1]}$, $H_0 = H$, such that $\forall\, F \in C_0(\mathbb{R})$, $(F(H_t))_t$ is a norm continuous curve in $\mathcal{A}$. If the gap does not close; $\forall\, t\,, E_{0t} < E_{1t}$, then $F_{(E_{0t},E_{1t})}(H_t)$ can always be defined and is a projection. Recall that if a projection p is homotopic to a projection q, then $[p]_0 = [q]_0$. So $\mathcal{L}(E_{0t}, E_{1t})$ is constant in t.

Concrete K_0-theoretic gap-labelling.

Suppose $\nu : K_0(\mathcal{A}) \to \mathbb{R}$ is a homomorphism of groups preserving order. Then $\nu \circ \mathcal{L} : \mathrm{Gap}(H) \to \mathbb{R}$ is also topologically stable: an example of a topologically quantized number!

Example.

$$\nu = \tau_* \qquad \tau_*[p]_0 = \tau(p)$$

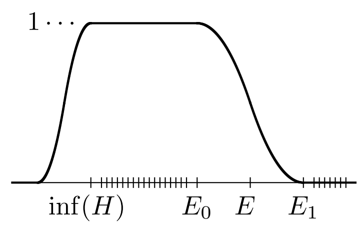
Recall that $\tau_* \circ \mathcal{L}((E_0, E_1)) = \mathrm{IDS}(E)$ for any $E_0 < E < E_1$ (stable position of the plateaus in the graphs).

14 (23 February 2009)

Gap labelling, cont.

Recall. Abstract GL. There is a positive map

$$\begin{aligned}\mathcal{L} :&\mathrm{Gap}(H) \to K_0(\mathcal{A})\\ &(E_0, E_1) \mapsto [F_{(E_0,E_1)}(H)]_0\end{aligned}$$



Gap labels are stable under perturbations which do not close the gap.

Concrete GL. $\nu : K_0(\mathcal{A}) \to \mathbb{R}$ is a positive group homomorphism. Then

$$\nu \circ \mathcal{L} : \mathrm{Gap}(H) \to \mathbb{R}$$

is a gap labelling by numbers.

In essentially all cases we have only one $\nu = c\tau_*$, $c > 0$, where τ is a trace on $\mathcal{A}$.

In many cases (unique ergodic, $\mathbb{R}^n$-invariant probability measure on $\Omega_{\mathcal{P}}$) there is a unique trace on $\mathcal{A}$ up to normalization. (In the case of unbounded operators, normalization normalization does not have meaning.)

Q: concrete gap labelling says that the values the IDS takes on gaps belong to $\tau_* K_0(\mathcal{A})$.

1) Given $r \in \tau_* K_0(\mathcal{A})$, is there a gap in $\sigma(H)$ such that r is a gap label for H? Difficult question.

   - Example: Harper model
     $H \in \mathcal{B}(\ell^2(\mathbb{Z}))$

$$H_\alpha = \underbrace{-\partial^2_{x,\mathrm{discrete}}}_{=\hat{p}^2 \text{ discr.}} + V_{\alpha,\theta} \quad \alpha, \theta \text{ angles}$$

The discrete version of the Laplacian:

$$\left(\left(-\partial^2_{x,\text{discrete}}\right)\psi\right)(n) = -(\psi(n+1) + \psi(n-1) - 2\psi(n))$$
$$V_{\alpha,\theta}(n) = 2\cos(2\pi n\alpha + \theta)$$

If $\alpha \notin \mathbb{Q}$, then $\sigma(H) =$ Cantor set. [Elliott, Choi, *et.al.*]

- Another example. Fibonacci.

$$H = -\partial^2_{x,\text{discrete}} + V \qquad \text{on } \mathbb{Z} = \{\ldots, \text{ababb}, \ldots\}$$
$$V(n) = \lambda \begin{cases} 1 & \text{if } n \text{ is on } a \\ 0 & \text{if } n \text{ is on } b \end{cases}$$

2) What is $\underbrace{\tau_* K_0(\mathcal{A})}_{\text{gap labelling group}}$?

Theorem (Bellisard, among others)**.** *If $\mathcal{P}$ is Delone of FLC and $\mathbb{P}$ is an invariant probability measure defining τ, then* [2]

$$\tau_* K_0(\mathcal{A}) = \mathbb{P}(C(\mathfrak{E}_{\mathcal{P}}, \mathbb{Z}))$$
$$\mathcal{P} \rightsquigarrow \Omega_{\mathcal{P}} = \overline{\{\mathcal{P} - x \mid x \in \mathbb{R}^n\}}$$
$$\cup$$
$$\mathfrak{E}_{\mathcal{P}} = \overline{\{\mathcal{P} - x \mid x \in \mathcal{P}\}} \quad \text{(totally disconnected and compact)}$$

(canonical transversal)

$$\mathbb{P}(f) = \int_{\mathfrak{E}_{\mathcal{P}}} f\,, \qquad f = \sum_{\substack{\text{finite} \\ U_i \text{ clopen}}} n_i \mathbf{1}_{U_i} \Rightarrow \mathbb{P}(f) = \sum n_i |U_i|$$

If C is an R-patch, that is, $C = \underbrace{B_R[\mathcal{P} - p]}_{(=B_R(0)\cap(\mathcal{P}-p))}$, $p \in \mathcal{P}$, then

$$U_C = \{Q \in \mathfrak{E}_{\mathcal{P}} \mid B_R(Q) = C\}$$

and if $\mathbb{P}$ is ergodic then $\mathbb{P}(U_C) =$ frequency of C in $\mathcal{P}$.

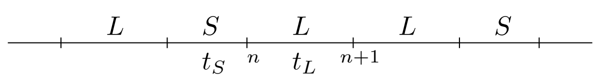
[2] Font substitute $\mathfrak{E}$ for the symbol used in the lecture.

15 (25 February 2009)

Graphics from the book *Quasicrystals: Structure and Physical Properties. (Notes for this section are incomplete.)*[3]

A 1-d example. Substitution:

$$\begin{aligned} L &\longmapsto LS \\ S &\longmapsto L \end{aligned}$$



$$H(\psi)(n) = t_{n+1}\psi(n+1) + t_{n-1}\psi(n-1) + \epsilon_n \psi(n)$$

$$\underset{\text{subscripts: vertex labels}}{t_{n,n+1}} = \begin{cases} t_{\mathrm{L}} \\ t_{\mathrm{S}} & \text{(set } t_S = 1 \text{ in what follows.)} \end{cases}$$

$$t_{n,n-1} = \bar{t}_{n-1,n} \quad (H \text{ is self-adjoint})$$

$$\epsilon_n = \begin{cases} \epsilon_{\mathrm{LL}} \\ \epsilon_{\mathrm{LS}} \\ \epsilon_{\mathrm{SL}} \end{cases}$$

all expressed in terms of the parameters $t_L, t_S, \epsilon_{LL}, \epsilon_{LS}, \epsilon_{SL}$.

Constant potential $\epsilon_{\mathrm{LL}} = \epsilon_{\mathrm{LS}} = \epsilon_{\mathrm{SL}} = 1$, and t_L the (running, or is it hopping?) parameter (maintaining $t_S = 1$), then when $t_L = 1$ the system is periodic with spectrum $[-2, 2] + 1$.

Potential $\epsilon_{\mathrm{LS}} = \epsilon_{\mathrm{SL}} = 1$ and $\epsilon_{\mathrm{LL}} = -1$, and t_L the parameter there is an associated graphic in the (energy $\times\, t_L$)-plane[4].

[3] *Quasicrystals: Structure and Physical Properties*, ed.by Hans-Rainer Trebin. Wiley 2003.
[4] See page 240 of *Quasicrystals: Structure and Physical Properties.*

Transport Coefficients (response theory)

Let H be the Hamiltonian of a system on $\mathcal{H}$. Recall equation of motion for A (linear operator on $\mathcal{H}$):

$$\begin{aligned}\dot{A} &= \frac{1}{i}[A, H] \quad (\hbar = 1)\\ &= i\mathcal{L}(A) \quad \text{where } \mathcal{L} \text{ is a derivation:}\\ \mathcal{L} &= [H, \cdot\,] \quad \mathcal{L} \text{ is the } \underline{\text{Liouvillian}}\end{aligned}$$

$$\begin{aligned}A(t) &= U(t, t_0)A(t_0)U(t, t_0)^{-1} \quad \text{can now be written}\\ A(t) &= \mathrm{e}^{it\mathcal{L}}(A(t_0)) \quad \text{(on an appropriate domain)}\end{aligned}$$

Suppose ρ is the density matrix of the system at equilibrium.

Ex: $\rho = \dfrac{\mathrm{e}^{-\beta H}}{2}$

In particular, equilibrium $\Longrightarrow \quad \dot{\rho} = 0 \Leftrightarrow [\rho, H] = 0$

Idea: perturb the dynamics slightly out of equilibrium such that (magically) the system evolves under the perturbed dynamics to a new equilibrium described by a density matrix ρ_{per} .

So we suppose that the perturbation is given by a derivation δ (vector-valued):

$$\begin{aligned}\mathcal{L}_{\text{per}} &= \mathcal{L} + \lambda \cdot \delta, \text{ where}\\ \delta &= (\delta_1, \ldots, \delta_f)\\ \lambda &= (\lambda_1, \ldots, \lambda_f)\\ \lambda \cdot \delta &= \sum_{I=1}^{f} \lambda_i \delta_i\end{aligned}$$

Hypothesis:

(1) $$\rho_{\text{per}} = \mathrm{C}\text{ - }\lim_{t\to\infty} \mathrm{e}^{it\mathcal{L}_{\text{per}}}(\rho)\,, \quad \text{where now}$$

$$\mathrm{C}\text{ - }\lim_{t\to\infty} x(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\mathrm{d}t \quad \text{(Cesàro)}$$

The system responds to the perturbation. How? The new time derivative:

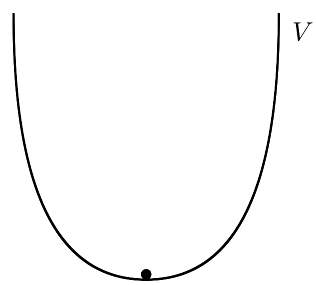
(2) $$\mathcal{L}_{\text{per}}(H) = [H, H] + \lambda \cdot \delta H = \lambda \cdot \delta H$$

(The new Hamiltonian is not the one describing energy; energy is constant.)

Ex: $n = 1\,,\ \delta = i[\hat{p},\,\cdot\,]$, this generates translation

$$\begin{aligned} \mathrm{e}^{\lambda\delta}(f)(x) &= f(x-\lambda) \quad \text{(on an appropriate domain)} \\ H &= \frac{\hat{p}^2}{2m} + V \\ \mathrm{e}^{it\mathcal{L}_{\text{per}}}(H) &= \frac{\hat{p}^2}{2m} + V(\,\cdot\, - \lambda t) \end{aligned}$$

[5]



Confining potential

(Only a constant potential would be invariant under translation.)

One expects that the new equilibrium involves a force which keeps the system in balance. This force is

$$\frac{1}{i}\delta H = V' \quad \text{(the derivative of } V\text{)}$$

called the *gradient force.*

Definition. δH is a *generalized force.*

Let us measure δH at the new equilibrium

$$\langle \delta H \rangle_{\text{per}} = \tau(\rho_{\text{per}} H) = F(\lambda)$$

F is called the *response function.* Suppose that $\tau \circ \delta = 0$ (translation invariant under the derivation). If A is a linear operator, then (allowing a bit of sloppiness regarding interchange of C-limits and operators)

$$\begin{aligned} \tau(\rho_{\text{per}} A) &\overset{\text{see (1)}}{=} \text{C - }\lim_{t\to\infty} \tau(\mathrm{e}^{it\mathcal{L}_{\text{per}}}(\rho)A) \\ \tau\left(\mathcal{L}_{\text{per}}(\rho)A\right) &= \tau(\lambda \cdot \delta(\rho)A) \\ &\overset{*}{=} \sum_i \lambda_i \left(\underset{=0}{\tau(\delta_i(\rho A))} - \tau(\rho\delta_i A) \right) \\ \tau(\rho_{\text{per}} A) &= \text{C - }\lim_{t\to\infty} \tau\left(\rho\, \mathrm{e}^{-it\mathcal{L}_{\text{per}}}(A)\right) \end{aligned}$$

[5]Where Taylor series converge everywhere.

(The equality marked $*$ uses the properties of the derivation δ.) [6]

Formal Dyson expansion:

$$\mathrm{e}^{-it\mathcal{L}_{\mathrm{per}}} = \mathrm{e}^{-it\mathcal{L}} + \sum_{\nu=1}^{\infty} \overbrace{\int_{\substack{s_1,\ldots,s_{\nu+1}\geq 0 \\ \sum_{i=1}^{\nu+1} s_i = 1}} \mathrm{e}^{-is_1\mathcal{L}} \lambda\delta \mathrm{e}^{-is_2\mathcal{L}} \cdots \mathrm{e}^{-is_{\nu+1}\mathcal{L}}}^{\mathcal{O}(\lambda^\nu)}$$

Definition. A *transport coefficient* is a coefficient in the formal Taylor expansion of $F(\lambda)$ around $\lambda = 0$.

We use the Dyson expansion to do this Taylor expansion.

0-order:

$$\tau\left(\mathrm{e}^{-it\mathcal{L}}(\rho)\delta H\right) = \tau(\rho\delta H)$$

Ex:

$$\delta H = i[B, H] \quad (B \text{ operates on } \mathcal{H})$$

Suppose ρBH is trace-class ($\Rightarrow H\rho B$ and ρHB are trace-class).

$$\frac{1}{2}\tau(\rho\delta H) = \tau(\rho(BH - HB)) = \underbrace{\tau(\rho BH)}_{\substack{=\tau(H\rho B) \\ =\tau(\rho HB)}} - \tau(\rho HB) = 0$$

so, practically, one looks only at 1^{st}-order coefficients.

[6] $\tau(\lambda \cdot \delta(\rho A)) = \sum_i \tau(\lambda_i \delta_i(\rho A)) = \sum_i \lambda_i \tau(\delta_i(\rho A)) = \sum_i \lambda_i \tau(\delta_i(\rho)A + \rho\delta_i A)$

16 (27 February 2009)

Transport coefficients, cont.

Recall:

$$\begin{array}{rll} \mathcal{L} = [H, \cdot\,] & \rho = \text{ equilibrium density matrix} & \\ \mathcal{L}_{\text{per}} = \mathcal{L} + \vec{\lambda}\cdot\vec{\delta}\,, & \vec{\delta} = (\delta_1, \ldots, \delta_n) \quad \text{commuting derivations:} & [\delta_i, \delta_j] = 0 \\ & \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \quad \text{complex constants} & \end{array}$$

Given the perturbation $\vec{\lambda}\cdot\vec{\delta}$ of $\mathcal{L}$ there is a response: $\langle \vec{\delta} H\rangle_{\text{per}} = \tau(\rho_{\text{per}}\vec{\delta}H)$, according to our philosophy.

Hypothesis:

$$\rho_{\text{per}} = \text{“}\underbrace{\lim_{t\to\infty}}_{\text{very often must be regularized}}\text{”}\underbrace{\mathrm{e}^{it\mathcal{L}_{\text{per}}}(\rho)}_{\text{equation of motion}}$$

Assumptions:

1) $\tau \circ \delta = 0$

   Then

$$\langle \vec{\delta} H\rangle_{\text{per}} = \tau(\rho_{\text{per}}\vec{\delta}H) = \lim_{t\to\infty} \tau(\mathrm{e}^{it\mathcal{L}_{\text{per}}}(\rho)\vec{\delta}\cdot\vec{H})$$

$$\text{(formally, } \tau(\mathcal{L}_{\text{per}}(\rho)A) = -\tau(\rho\mathcal{L}_{\text{per}}(A)))$$

$$(*) \qquad = \lim_{t\to\infty} \tau(\rho\mathrm{e}^{-it\mathcal{L}_{\text{per}}}\vec{\delta}\cdot\vec{H})$$

2) Bold assumption: use the Dyson expansion for $\mathrm{e}^{-it\mathcal{L}_{\text{per}}}$

$$(*) \overset{\text{Dyson}}{\underset{\text{power series in } \lambda}{=}} \underbrace{\tau(\rho\,\vec{\delta}\cdot\vec{H})}_{\lambda^0\text{-term; mostly } = 0} + \underbrace{\lim_{t\to\infty} \tau\left(\rho \int_{\substack{s_0, s_1 \geq 0 \\ s_0+s_1=1}} \mathrm{e}^{-is_0\mathcal{L}}\vec{\lambda}\cdot\vec{\delta}\mathrm{e}^{-is_1\mathcal{L}}\vec{\delta}\vec{H}\right)}_{1^{\text{st}}\text{ order term}} + \mathcal{O}(\lambda^2)$$

translate this thing

3) 1^{st} order term. Assumption: $\mathcal{O}(\lambda^2)$ is really negligible.

   $\langle\vec{\delta}H\rangle_{\text{per}} = \sigma^\delta\vec{\lambda}$ (σ^δ is a matrix; a tensor in the case of the higher order terms.)

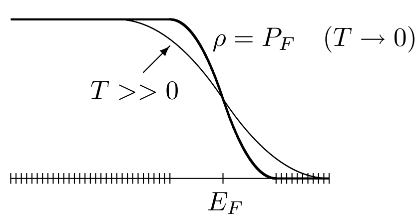
   where

$$\sigma^\delta_{\nu\mu} = \lim_{t\to\infty} \overbrace{\tau\left(\rho\int_0^t \mathrm{e}^{-i(t-s_1)\mathcal{L}}\delta_\mu \mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\right)\mathrm{d}s_1}^{\sigma_{\nu\mu}(t)}$$

   is the tensor of transport coefficients in the 1^{st} order approximation (higher orders; higher order tensors). We have:

$$\begin{aligned}\sigma_{\nu\mu}(t) &= \tau\left(\underbrace{\mathrm{e}^{i(t-s_1)\mathcal{L}}(\rho)}_{=\rho\text{ by invariance}}\ \delta_\mu\int_0^t \mathrm{e}^{-s_1\mathcal{L}}\delta_\nu H\mathrm{d}s_1\right)\\ &\overset{7}{=}\tau\left(-\delta_\mu(\rho)\int_0^t \mathrm{e}^{-is\mathcal{L}}\delta_\nu H\mathrm{d}s\right)\end{aligned}$$

Now *a priori* there is a singularity if $t\to\infty$ in case $\delta_\nu\notin \operatorname{Ker}\mathcal{L}^\perp$. Suppose that the temperature T is very low and the Fermi energy $E_F\in\operatorname{Gap}(H)$. Hence $\rho = P_F =$ the spectral projection of H to states below the Fermi energy:



Remark. If $p = p^2$ and δ is any derivation:

$$\begin{aligned}\delta(p^2) &= p\delta(p)+\delta(p)p\,,\ \text{and}\\ &||\\ \delta(p) &= p\delta(p)+p^\perp\delta(p)\quad (p^\perp = 1-p)\\ &||\\ \delta(p) &= \delta(p)p+\delta(p)p^\perp\quad(\text{ditto})\\ \Leftrightarrow \delta(p)p &= p^\perp\delta(p)\ \text{and}\ p\delta(p)=\delta(p)p^\perp\end{aligned}$$

$(*)$ put this together $\overset{8}{\Rightarrow}\delta(p) = p\delta(p)p^\perp + p^\perp\delta(p)p$

[7] Using the fact that δ_μ is a derivation and that τ is invariant under δ.

The result marked $(*)$ is an extremely important algebraic calculation.

Suppose that we have an eigenbasis of H by $\{\psi_i\}_i$ (perhaps generalized eigenvectors). Then

$$\tau\left(\delta_\mu(P_F)\int_0^t \mathrm{e}^{-is\mathcal{L}}\delta_\nu H \mathrm{d}s\right) = \sum_{i,j} \underbrace{\langle\psi_i \,|\, \delta_\mu P_F \,|\, \psi_j\rangle}_{X_{ij}} \langle\psi_j \,|\, \int_0^t \mathrm{e}^{-is\mathcal{L}}\delta_\nu H \,|\, \psi_i\rangle$$

where $X_{ij} = 0$ if both ψ_i and ψ_j belong to either $\operatorname{Im} P_F$ or its orthogonal complement

Notes are incomplete from here,

$$\langle\psi_j \,|\, H\delta_\nu H \,|\, \psi_i\rangle - \langle\psi_j \,|\, \delta_\nu(H)H \,|\, \psi_i\rangle$$

$$H|\psi_i\rangle = E_i|\psi_i\rangle$$

$$\Rightarrow |E_i - E_j| \geq |\tilde{E}_1 - \tilde{E}_0|$$

to here

So

$$\langle\psi_j| \int_0^t \mathrm{e}^{-is\mathcal{L}}\delta_\nu|\psi_i\rangle \mathrm{d}s = \int_0^t \mathrm{e}^{-is(E_j-E_i)}\mathrm{d}s \,\langle\psi_j \,|\, \delta_\nu H \,|\, \psi_i\rangle$$

$$\begin{aligned}\lim_{t\to\infty}\int_0^t \mathrm{e}^{-is\omega}\mathrm{d}s &= \text{FT of the Heaviside function}\\ &= \underbrace{PV\left(\frac{1}{i\omega}\right)}_{\text{principal value}} + \frac{\pi}{2}\delta(\omega) \text{ in the distribution space}\end{aligned}$$

Hence

$$\frac{1}{i(E_j - E_i)}\langle\psi_j \,|\, \delta_\nu H \,|\, \psi_i\rangle = \frac{1}{i}\langle\psi_j \,|\, \mathcal{L}^{-1}\delta_\nu H \,|\, \psi_i\rangle$$

Drawing everything together,

Proposition. $\sigma^\delta_{\nu\mu} = -\frac{1}{i}\tau\left(\delta_\mu(P_F)\underset{\substack{| \\ \text{not at all invertible so need } P_F}}{\mathcal{L}^{-1}}\delta_\nu H\right)$

[8] $\delta(p) = p(p\delta(p)) + p^\perp(p^\perp\delta(p)) = p\delta(p)p^\perp + p^\perp\delta(p)p$

Using the extremely important $(*)$ twice (as well as commuting properties of $\mathcal{L}$ with the projections),

$$-\frac{1}{i}\tau\left(\delta_\mu(P_F)\mathcal{L}^{-1}\delta_\nu H\right) = -\frac{1}{i}\tau\left(P_F\delta_\mu(P_F)P_F^\perp\mathcal{L}^{-1}\delta_\nu H + P_F^\perp\delta_\mu(P_F)P_F\mathcal{L}^{-1}\delta_\nu H\right)$$
$$= -\frac{1}{i}\tau\left(P_F\delta_\mu(P_F)P_F^\perp\mathcal{L}^{-1}(\delta_\nu(H))P_F + P_F^\perp\delta_\mu(P_F)P_F\mathcal{L}^{-1}(\delta_\nu(H))P_F^\perp\right)$$

Lemma.

$$P_F^\perp\mathcal{L}^{-1}(\delta_\nu(H))P_F = -P_F^\perp\delta_\nu(P_F)P_F$$
$$\text{and } P_F\mathcal{L}^{-1}(\delta_\nu(H))P_F^\perp = P_F\delta_\nu(P_F)P_F^\perp$$

Proof. Apply $\mathcal{L}$ to right-hand side:

$$\mathcal{L}\left(P_F^\perp\delta_\nu(P_F)P_F\right) = P_F^\perp\underbrace{[H,\delta_\nu(P_F)]}_{\substack{\delta_\nu[H,P_F]-[\delta_\nu(H),P_F] \\ =0-[\delta_\nu(H),P_F]}}P_F \text{ (commutator in } H \text{ commutes with } (\delta_\nu H))$$
$$= -P_F^\perp[\delta_\nu(H),P_F]P_F$$
$$= -P_F^\perp\delta_\nu(H)P_F$$

This proves the first equality. The second is proved in the same sort of way. □

Final result:

Proposition. $\sigma^\delta_{\nu\mu} = -i\tau\left(P_F\delta_\mu(P_F)\delta_\nu(P_F) - \delta_\nu(P_F)\delta_\mu(P_F)\right)$

This is a non-commutative Chern character !!!

17 (2 March 2009)

Transport coefficients, cont.

<u>Result</u>: 1^{st}-order response to a perturbation $\mathcal{L}_{\text{per}} = \mathcal{L} + \vec{\lambda}\cdot\vec{\delta}$, where $\vec{\delta} = (\delta_1, \ldots, \delta_n)$ are commuting derivations and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ are complex constants.

Then
$$\langle \delta_\nu H\rangle_{\text{per}} = \sum_\mu \sigma^\delta_{\nu,\mu}\lambda_\mu + \mathcal{O}(\lambda^2) \quad (0^{\text{th}}\text{-order response} = 0)$$

defines the first order response (transport) coefficients

$$\sigma^\delta_{\nu\mu} = \text{C-}\lim_{t\to\infty} \tau\left(\rho \int_{\substack{s_0,s_1\,\geq\,0\\ s_0+s_1\,=t}} \underbrace{\mathrm{e}^{-is_0\mathcal{L}}}_{\text{(put this onto } \rho\text{, but } \rho \text{ is invariant so cross this term off)}} \delta_\mu \mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\, \mathrm{d}s_0\mathrm{d}s_1\right)$$

$$= \text{C-}\lim_{t\to\infty} \tau\left(\int_{\substack{s_0,s_1\,\geq\,0\\ s_0+s_1\,=t}} \underbrace{\left(\mathrm{e}^{-is_0\mathcal{L}}(\rho)\right)}_{\rho} \delta_\mu \mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\, \mathrm{d}s_0\mathrm{d}s_1\right)$$

$$= \text{C-}\lim_{t\to\infty} \tau\left(\delta_\mu \int_{\substack{s_0,s_1\,\geq\,0\\ s_0+s_1\,=t}} \mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\, \mathrm{d}s_0\mathrm{d}s_1\right)$$

$$= \underbrace{0}_{\text{because } \tau \text{ is invariant under derivation } (\tau\circ\delta=0)\text{, then } \tau(\delta_\mu(\rho\int \text{etc. }))=0} - \text{C-}\lim_{t\to\infty} \tau\left((\delta_\mu(\rho))\int_0^t \mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\, \mathrm{d}s_1\right)$$

$$= -\tau\left(\delta_\mu(\rho)\underbrace{\mathcal{D}(\mathcal{L})}_{\lim_{t\to\infty}\int_0^t \mathrm{e}^{-is\omega}\mathrm{d}s \overset{\text{dist'n}}{=} \text{PV}\frac{1}{i\omega}+\frac{\pi}{2}\delta(\omega)=:\mathcal{D}(\omega)}\delta_\nu(H)\right) \text{ (which may be infinite)}$$

Attention if $\delta_\nu(H)$ is not $\perp$ to $\ker\mathcal{L}$.

Assumption $T \searrow 0$ and $\underset{\text{Fermi energy}}{E_F} \in \text{Gap}(H)$

Mathematical assumption is $\rho = P_F =$ the spectral projection of H onto states $\leq E_F$. This avoids the singularity.

Suppose we have $\{\psi_i\}_i$ an eigenbasis of H (in the generalized sense).

So $H\psi_i = E_i\psi_i$ (that is, $H\,|\,\psi_i\rangle = E_i\,|\,\psi_i\rangle$).

$$\langle\psi_i\,|\,\underset{\mathcal{L}(\,\cdot\,)=[H,\,\cdot\,]}{\mathcal{L}}(\delta_\nu H)\,|\,\psi_j\rangle = \langle\psi_i\,|\,(H\delta_\nu H - (\delta_\nu H)H)\,|\,\psi_j\rangle$$

$$= (E_i - E_j)\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle$$

So,

$$\begin{aligned}\langle\psi_i\,|\,\mathcal{L}^n(\delta_\nu H)\,|\,\psi_j\rangle &= \langle\psi_i\,|\,\mathcal{L}(\mathcal{L}^{n-1}(\delta_\nu H))\,|\,\psi_j\rangle \\ &= (E_i - E_j)\langle\psi_i\,|\,\mathcal{L}^{n-1}(\delta_\nu H)\,|\,\psi_j\rangle\end{aligned}$$

Hence

$$\langle\psi_i\,|\,\mathrm{e}^{-is_1\mathcal{L}}\delta_\nu H\,|\,\psi_j\rangle = \mathrm{e}^{-is_1(E_i-E_j)}\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle$$

So we need that $E_i \neq E_j$ (to avoid $\omega = 0$ in the distribution $\mathcal{D}(\omega)$). In that case

$$\langle\psi_i\,|\,\lim_{t\to\infty}\int_0^t \mathrm{e}^{-is_1\mathcal{L}}\mathrm{d}s_1\delta_\nu H\,|\,\psi_j\rangle = \frac{1}{i(E_i - E_j)}\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle$$

Since $\rho = P_F$ is a projection and δ_μ is a derivation (using the beautiful formula $(*)$ from the previous lecture)

$$\delta_\mu P_F = P_F(\delta_\mu P_F)P_F^\perp + P_F^\perp(\delta_\mu P_F)P_F$$

Now

$$\langle\psi_i\,|\,\delta_\nu H\,|\,\psi_j\rangle \neq 0 \Leftrightarrow ((E_i \leq E_F)\wedge(E_j > E_F))\vee((E_j \leq E_F)\wedge(E_i > E_F))$$

Since $E_F \in (\tilde{E}_0, \tilde{E}_1)$, a gap in $\sigma(H)$, we have

$$|E_i - E_j| \geq |\tilde{E}_1 - \tilde{E}_0|$$

As a consequence,

$$\begin{aligned}\sigma^\delta_{\nu\mu} &= -\tau\,(\delta_\mu(P_F)\underset{\substack{\vee \\ 1=P_F+P_F^\perp}}{\mathcal{D}}(\mathcal{L})\delta_\nu(H)) \\ &= -\tau\,(P_F\delta_\mu(P_F)(\mathcal{D}\mathcal{L})\delta_\nu(H)) - \tau\left(P_F^\perp\delta_\mu(P_F)\mathcal{D}(\mathcal{L})\delta_\nu(H)\right) \\ &= -\tau\,(P_F\delta_\mu(P_F)(\mathcal{D}(\mathcal{L})\delta_\nu(H))P_F) - \tau\left(P_F^\perp\delta_\mu(P_F)(\mathcal{D}(\mathcal{L})\delta_\nu(H))P_F^\perp\right) \\ &= -\tau\left(P_F\delta_\mu(P_F)P_F^\perp\frac{1}{i}\mathcal{L}^{-1}(\delta_\nu H)P_F\right) - \tau\left(P_F^\perp\delta_\mu(P_F)P_F\frac{1}{i}\mathcal{L}^{-1}(\delta_\nu H)P_F^\perp\right) \\ &\overset{\text{claim}}{=} -i\tau\,(P_F\delta_\mu(P_F)\delta_\nu(P_F) - \delta_\nu(P_F)\delta_\mu(P_F)) \\ &\overset{\text{or}}{=} -i\tau\left(P_F\delta_\mu(P_F)P_F^\perp\delta_\nu(P_F)P_F - \delta_\nu(P_F)\delta_\mu(P_F)\right)\end{aligned}$$

In other words the claim is

$$\begin{aligned} P_F^{\perp}\mathcal{L}^{-1}(\delta_\nu H)P_F &= -P_F^{\perp}(\delta_\nu P_F)P_F \quad \text{and} \\ P_F\mathcal{L}^{-1}(\delta_\nu H)P_F^{\perp} &= P_F(\delta_\nu P_F)P_F^{\perp} \end{aligned}$$

Proof. Hit both sides of the first equation with $\mathcal{L}$ ($\mathcal{L}$ leaves P_F invariant).

$$\begin{aligned} \text{left-hand side: } \mathcal{L}\left(P_F^{\perp}\mathcal{L}^{-1}(\delta_\nu H)P_F\right) &= P_F^{\perp}(\delta_\nu H)P_F \\ \text{right-hand side: } \mathcal{L}\left(-P_F^{\perp}(\delta_\nu P_F)P_F\right) &= -P_F^{\perp}[H,\delta_\nu P_F]P_F \\ & \qquad\quad \mathcal{L}(\,\cdot\,)=[H,\,\cdot\,] \\ &= -P_F^{\perp}\left(\delta_\nu\left([H,P_F]\right)-[\delta_\nu H,P_F]\right)P_F \\ & \qquad\qquad\quad [H,P_F]=0 \\ &= P_F^{\perp}\left((\delta_\nu H)P_F - P_F(\delta_\nu H)\right)P_F \\ &= P_F^{\perp}(\delta_\nu H)P_F \quad \text{(QED first equation)} \end{aligned}$$

The proof of the second equation is similar and picks up an extra minus sign. □

So this is the result

$$\sigma^{\delta}_{\nu\mu} = i\,\tau(P_F\left[(\delta_\nu P_F),(\delta_\mu P_F)\right])$$

Consequence: *$\sigma^{\delta}_{\nu\mu}$ is a topological invariant.*

Remark. $\sigma^{\delta}_{\nu\mu}$ is anti-symmetric.

Example. QHE in $\mathbb{R}^2$ (the quantum Hall effect).

$$\begin{aligned} \delta &= (\delta_1.\delta_2) \qquad & \delta_\nu &= [\hat{q}_\nu,\,\cdot\,] \\ \lambda &= (\lambda_1,\lambda_2) \qquad & \lambda_\nu &= e\,E_\nu \end{aligned}$$

e is the electric charge

E_ν is the external electric field

18 (4 March 2009)

Cyclic cohomology and higher traces.

To describe topologically quantized transport coefficients we will use $\underset{\text{algebra}}{\mathcal{A}}$, $\underset{\text{trace}}{\tau}$, $\underset{\text{comm. derivns on } \mathcal{A}}{\vec{\delta} = (\delta_1, \dots, \delta_n)}$ such that $\tau \circ \delta_\nu = 0\ \forall \nu$.

This is the realm of *higher traces* on Banach algebras.

Let $\mathcal{B}$ be an associative algebra over k (field). Let $C^n_\lambda(\mathcal{B}) = \{\text{cyclic } n{+}1\text{-forms}\}$. So $\eta \in C^n_\lambda(\mathcal{B})$ this is a map

$$\eta : \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{n+1} \to \mathsf{k}$$

which is linear in each argument and *cyclic*:

$$\eta(A_0, \dots, A_n) = (-1)^n \eta(A_1, \dots, A_n, A_0)\,, \quad A_i \in \mathcal{B}$$

$C^n_\lambda(\mathcal{B})$ is a VS over k.

Define a differential operator

$$\begin{aligned} \mathtt{b} : C^n_\lambda(\mathcal{B}) &\to C^{n+1}_\lambda(\mathcal{B})\,, \text{where} \\ \eta \in C^n_\lambda(\mathcal{B}) \longmapsto & \\ \mathtt{b}\eta(A_0, \dots, A_{n+1}) &= \sum_{i=0}^{n} (-1)^i \eta(A_0, \dots, A_i \cdot A_{i+1}, \dots, A_{n+1}) \\ &\quad + (-1)^{n+1} \eta(A_{n+1} \cdot A_0, A_1, \dots, A_n) \end{aligned}$$

Lemma (Exercise)**.** $\mathtt{b} \circ \mathtt{b} = 0$. *The differential complex* $(C^n_\lambda(\mathcal{B}), \mathtt{b})$ *is a subcomplex (because of cyclic) of the complex for the Hochschild cohomology* $H(\mathcal{B}, \mathcal{B}^*)$ *(of* $\mathcal{B}$ *with coefficients in* $\mathcal{B}^*$*).*

Definition. *Cyclic cohomology of* $\mathcal{B}$ is the cohomology of $(C^n_\lambda(\mathcal{B}), \mathtt{b})$, denoted $H^nC(\mathcal{B})$. $\eta \in C^n_\lambda(\mathcal{B})$ is a *cyclic cocycle* if $\eta \in \operatorname{Ker} \mathtt{b}$.

Ex 1: $\mathcal{B}$ a C*-algebra and τ a bounded trace on $\mathcal{B}$.

$$\tau \in C^0_\lambda(\mathcal{B}) \Rightarrow \mathtt{b}\tau(A_0, A_1) = \tau(A_0 A_1) - \tau(A_1 A_0) = 0$$

since traces are cyclic. So a bounded trace is a cyclic 0-cocycle.

Ex 2: $\mathcal{M}$ an n-dim smooth, compact manifold without boundary, $\mathcal{B} = C^\infty(\mathcal{M})$ and for $\tau \in C^n_\lambda(\mathcal{B})$ (d is the exterior derivative on $\mathcal{M}$):

$$\tau(f_0, \dots, f_n) := \int_{\mathcal{M}} f_0 \,\mathrm{d} f_1 \,\mathrm{d} f_2 \cdots \mathrm{d} f_n \quad ((n+1)\text{-linear form})$$

$$
\begin{aligned}
f_0 \,\mathrm{d} f_1 \cdots \mathrm{d} f_n &= (-1)^{n-1} \underbrace{(\mathrm{d} f_n) f_0}_{\mathrm{d}(f_n f_0) - f_n \,\mathrm{d} f_0} \,\mathrm{d} f_1 \cdots \mathrm{d} f_{n-1} \text{ (sign from exterior product)} \\
&= (-1)^{n-1} \mathrm{d}(f_n f_0) \,\mathrm{d} f_1 \cdots \mathrm{d} f_{n-1} + (-1)^n f_n \,\mathrm{d} f_0 \cdots \mathrm{d} f_{n-1}
\end{aligned}
$$

$$\Longrightarrow$$

$$
\int_{\mathcal{M}} f_0 \,\mathrm{d} f_1 \cdots \mathrm{d} f_n = \underset{\substack{| \\ \mathrm{d}(f_n f_0)\,\mathrm{d} f_1 \cdots \mathrm{d} f_{n-1} \text{ is exact and there is no boundary}}}{0} + (-1)^n \int_{\mathcal{M}} f_n \,\mathrm{d} f_0 \cdots \mathrm{d} f_{n-1}
$$

So τ is cyclic.

Now

$$
\begin{aligned}
\mathtt{b}\tau(f_0, \ldots, f_{n+1}) &= \sum_{i=0}^{n} (-1)^i \tau(f_0, \ldots, f_i f_{i+1}, \ldots, f_{n+1}) \\
&\quad + (-1)^{n+1} \tau(f_{n+1} f_0, f_1, \ldots, f_n) \\
&= \int_{\mathcal{M}} (f_0 f_1) \,\mathrm{d} f_2 \,\mathrm{d} f_3 \cdots \mathrm{d} f_{n+1} - \int_{\mathcal{M}} f_0 \,\mathrm{d}(f_1 f_2) \,\mathrm{d} f_3 \cdots \mathrm{d} f_{n+1} \\
&\quad + \int_{\mathcal{M}} f_0 \,\mathrm{d} f_1 \,\mathrm{d}(f_2 f_3) \cdots \mathrm{d} f_{n+1} - \cdots \\
&\quad + (-1)^n \int_{\mathcal{M}} f_0 \,\mathrm{d} f_1 \,\mathrm{d} f_2 \cdots \mathrm{d}(f_n f_{n+1}) + (-1)^{n+1} \int_{\mathcal{M}} (f_{n+1} f_0) \,\mathrm{d} f_1 \,\mathrm{d} f_2 \cdots \mathrm{d} f_n
\end{aligned}
$$

and from this expansion it follows that $\mathtt{b}\tau(f_0, \ldots, f_{n+1}) = 0$ as follows: in each integral except the first and last expand $\mathrm{d}(f_i f_{i+1}) = f_i \,\mathrm{d} f_{i+1} + (\mathrm{d} f_i) f_{i+1}$; the resulting $2n+2$ integrals cancel in pairs (exercise).

So, τ *is a cyclic n-cocycle*

Result: almost, $HC^n\left(C^\infty(\mathcal{M})\right) = H^n_{deRahm}(\mathcal{M})$.

Little catch: $C^\infty(\mathcal{M})$ is not a C*-algebra.

Reassuring: de Rahm cohomology $\cong$ Čech-cohomology and therefore is purely topological.

The pairing with K-theory

A cyclic cocycle over a C*-algebra $\mathcal{A}$ (over $\mathbb{C}$) defines a homomorphism

$$K_0(\mathcal{A}) \to \mathbb{C}$$

Suppose $\mathcal{A}$ is unital. Recall that $K_0(\mathcal{A})$ is made from homotopy classes of projections in $M_k(\mathcal{A})\,\forall k$. First extend a cyclic cocycle η of $\mathcal{A}$ to one, $\tilde{\eta}$, on $M_k(\mathcal{A})$. Let Tr be the standard trace on $M_k(\mathbb{C})$. Then, with $M_i \in M_k(\mathbb{C})$ and $A_i \in \mathcal{A}$, define

$$\tilde{\eta}(M_0 \otimes A_0, M_1 \otimes A_1, \ldots, M_n \otimes A_n) = \operatorname{Tr}(M_0 M_1 \cdots M_n)\eta(A_0, \ldots, A_n)$$

with linear extension to the general element of $M_k(\mathcal{A})$. Then $\tilde{\eta}$ is cyclic.

Proposition. *Let p and q be homotopic projections in $M_k(\mathcal{A})$, and let η be a cyclic n-cocycle. Then*

$$\tilde{\eta}(\underbrace{p, p, \ldots, p}_{n+1}) = \tilde{\eta}(\underbrace{q, q, \ldots, q}_{n+1})$$

Remark (Marcy). If n is odd then, since $\tilde{\eta}$ is cyclic, $\tilde{n}(p, \ldots, p) = -\tilde{n}(p, \ldots, p)$, and therefore $\tilde{n}(p, \ldots, p) = 0$.

Proof of the Proposition. Suppose $n = 2m$ and let $m = 1$ (other values of m are left as an exercise). Suppose there is a differentiable homotopy $p(t)$ between p and q, so that $p(0) = p$ and $p(1) = q$.

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\eta(p(t), \ldots, p(t)) = \eta(\dot{p}, p, \ldots, p) + \eta(p, \dot{p}, \ldots, p) + \cdots = (n+1)\eta(\dot{p}, p, \ldots, p)$$

From the 27 Feb. lecture,

$$\dot{p} = p\dot{p}p^{\perp} + p^{\perp}\dot{p}p$$

Now for $m = 1$,

$$\begin{aligned} \eta(p\dot{p}p^{\perp}, p, p) = {} & \underbrace{\mathtt{b}\eta(p\dot{p}, p^{\perp}, p, p)}_{=0 \text{ by cocycle hypoth.}} + \underbrace{\eta(p\dot{p}, p^{\perp}p, p)}_{p^{\perp}p=0} \\ & \underbrace{-\eta(p\dot{p}, p^{\perp}, p^2) + \eta(p^2\dot{p}, p^{\perp}, p)}_{\text{cancel because } p^2=p} \\ & = 0 \end{aligned}$$

And a similar argument shows that $\eta(p^{\perp}\dot{p}p, p, p) = 0$, so $\eta(\dot{p}, p, p) = 0$, and therefore $\eta(p(t), \ldots, p(t))$ is constant. $\square$

This allows us to define η and $\tilde{\eta}$ on $K_0(\mathcal{A})$.

19 (6 March 2009)

Cyclic cohomology and higher traces, cont.

If η is a cyclic cocycle

$$\langle \eta \,|\, [p]_0 \rangle = c_n \eta\overbrace{(p, \dots, p)}^{n+1}, \quad c_n = c_{2k} = \frac{1}{(2\pi i)^k} \frac{1}{k!}$$

is well-defined.

Aside (exercise): if $\eta = \mathtt{b}\tau$, then $\langle \mathtt{b}\tau \,|\, [p]_0 \rangle = 0$.

So one has a bilinear map $HC^n(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}$.

Ex.: $HC^n(\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

Smells like Bott periodicity.

Theorem. *If $\mathcal{A}$ is a C^*-algebra, then* $HC^n(\mathcal{A}) = \begin{cases} \textit{bounded traces on } \mathcal{A} & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$

Nothing interesting for C*-algebras.

So we need to include smoothness.

Let $\mathcal{A} = C(\mathcal{M})$; $\mathcal{M}$ an even dimensional, smooth, compact manifold without boundary.

$$\mathcal{A}^\infty = C^\infty(\mathcal{M}) \quad \text{in its Fréchet topology}$$

Require cyclic cocycles to be continuous w.r.t this topology.

Lemma. *For any projection $p \in \mathsf{M}_n(C(\mathcal{M}))$ there exists $\tilde{p} \in \mathsf{M}_n(C^\infty(\mathcal{M}))$ such that $p \sim \tilde{p}$.*

Then one could consider cyclic cocycles of $C^\infty(\mathcal{M})$ (in particular, C^∞-continuous) and extend the pairing with $K_0(C^\infty(\mathcal{M}))$ to a pairing with $K_0(C(\mathcal{M}))$

Purpose: to construct cyclic cocycles (continuous) for a dense subalgebra $\mathcal{A}^\infty$ of $\mathcal{A}$ such that the pairing extends from $K_0(\mathcal{A}^\infty)$ to $K_0(\mathcal{A})$.

Cycles. Look at $(\Omega, d, \int)$, a *cycle over a Banach algebra $\mathcal{B}$* of dimension n:

$$\Omega \text{ a graded algebra, } \Omega = \bigoplus_{i\in\mathbb{N}_0} \Omega^i,\ \Omega^i\Omega^j \subset \Omega^{i+j},\ \Omega^{n+p} = 0\ \forall\, p > 0\,;$$

$$d \text{ a differential of degree } +1\,, d^2 = 0\ +\ \text{Leibniz rule;}$$

$$\int \text{ is a } \textit{graded} \text{ trace[9]on } \Omega^n\,,\ \text{that is, } \int \text{ is linear and}$$

$$\int \omega_1\omega_2 = (-1)^{k(n-k)} \int \omega_2\omega_1\,,\ \omega_1 \in \Omega^k\,,\ \omega_2 \in \Omega^{n-k}\,;$$

$$\text{and lastly, } \mathcal{B} \text{ is a subalgebra of } \Omega^0\,.$$

Ex.: $\mathcal{B} = C^\infty(\mathcal{M})$, $\Omega = \Omega(\mathcal{M})$ the algebra of exterior forms over $\mathcal{M}$, d the exterior derivative, and $\int$ the integral of $(\dim\mathcal{M})$-forms over $\mathcal{M}$.

Proposition (Connes)**.** *Any cycle over $\mathcal{B}$ of dimension n defines a cyclic n-cocycle η (the character of the cycle):*

$$\eta(A_0, \dots, A_n) = \int A_0 dA_1 dA_2 \cdots dA_n$$

Conversely, any cyclic n-cocycle arises in this way.

Definition. An *n-trace* on a Banach algebra $\mathcal{B}$ is the character of a cycle of dimension n, $(\Omega', d, \int)$, over a dense subalgebra $\mathcal{B}'$ of $\mathcal{B}$, such that $\forall\, A_1, \dots, A_n \in \mathcal{B}'$, $\exists\, C(A_1, \dots, A_n)$ for which

$$\int (X_1 dA_1)(X_2 dA_2) \cdots (X_n dA_n) \le C\|X_1\|\|X_2\| \cdots \|X_n\|\,,\ \forall\, X_i \in \mathcal{B}'$$

This means that $\forall\, A_1, \dots, A_n \in \mathcal{B}'$

$$\underbrace{\mathcal{B}' \times \cdots \times \mathcal{B}'}_{\text{n}} \to \mathbb{C} \quad n\text{-linear map}$$

$$(X_1, \dots, X_n) \mapsto \int (X_1 dA_1)(X_2 dA_2) \cdots (X_n dA_n)$$

is bounded with norm $p(A_1, \dots, A_n) :=$ the smallest $C(A_1, \dots, A_n)$.

Theorem (Connes)**.** *Any n-trace on $\mathcal{B}$ extends to an algebra $\mathcal{B}''$, $\mathcal{B}' \subset \mathcal{B}'' \subset \mathcal{B}$, such that the inclusion $\mathcal{B}'' \xrightarrow{i} \mathcal{B}$ induces an isomorphism i_* in K-theory: $i_* : K_i(\mathcal{B}'') \underset{\cong}{\to} K_i(\mathcal{B})$.*

Consequence: An n-trace defines a functional on $K_i(\mathcal{B})$; first on $K_i(\mathcal{B}'')$ by continuous extension, and then by selecting for a dense-in-$\mathcal{B}$ representation of $\mathcal{B}''$.

[9] In addition, $\int$ is *closed:* $\int d\omega = 0$ for $\omega \in \Omega^{n-1}$. See the notes for 9 March.

Example.

$$
\begin{array}{lll}
\mathcal{A} = \text{ a C*-algebra,} & \tau = \text{ a trace,} & \delta = \text{ a derivation} \\
\updownarrow & \updownarrow^{\text{unbounded}} & \updownarrow^{\text{unbounded}} \\
\Omega = \mathcal{A} \otimes \bigwedge \mathbb{C}^2 & \displaystyle\int & d
\end{array}
$$

20 (9 March 2009)

Higher traces, cont.

Recall. A higher trace on a Banach algebra $\mathcal{B}$ is the character of a cycle $(\Omega, d, \int)$ on a dense sub-algebra $\mathcal{B}'$, which satisfies certain continuity conditions. These conditions allow us to extend (by continuity) the character to another sub-algebra $\mathcal{B}''$, $\mathcal{B}' \subset \mathcal{B}'' \subset \mathcal{B}$, such that the inclusion $\imath : \mathcal{B}'' \hookrightarrow \mathcal{B}$ induces an isomorphism $\imath_* : K_i(\mathcal{B}'') \to K_i(\mathcal{B})$. This ensures that the character defines a functional on $K_i(\mathcal{B})$: if $i = 0$ and the cycle has dimension n, even, then

$$[p]_0 \mapsto \int p \underbrace{dp \cdots dp}_{n}$$

Recall: a cycle $(\Omega, d, \int)$ over $\mathcal{B}$ of dimension n is:

- $\Omega = \bigoplus_{i \in \mathbb{N}_0} \Omega^i$ and $\Omega^{n+p} = 0$ for $p > 0$
- $\mathcal{B}$ is a sub-algebra of Ω^0
- d is a differential of degree 1, and $d^2 = 0$
- $\int$ is a graded trace which is closed:

graded: $\int \omega_1 \omega_2 = (-1)^{k(n-k)} \int \omega_2 \omega_1$, $\omega_1 \in \Omega^k$, $\omega_2 \in \Omega^{n-k}$
closed: $\int d\omega = 0$ for $\omega \in \Omega^{n-1}$ (omitted in the 6 March notes)

Terminology: a cycle over $\mathcal{B}'$ (dense sub-algebra of $\mathcal{B}$) is also called an unbounded cycle for $\mathcal{B}$. (So, $\mathcal{B} \subset \Omega^0$ but d and $\int$ have dense domain.)

Ex 1): $\mathcal{B}$ is a Banach algebra and τ an (unbounded) trace over $\mathcal{B}$. (τ is cyclic: $\tau(A_1 A_2) = \tau(A_2 A_1)$ if either A_1 or A_2 is trace class.) This is an (unbounded) 0-trace: namely the character of the 0-cycle

$$\Omega = \mathcal{B} \text{ (so } \Omega^n = 0 \; \forall \, n > 0), \; d = 0, \int = \tau.$$

Ex 2): $\mathcal{B} = C(\mathcal{M})$, $\mathcal{M}$ a compact (orientable as needed) n-dimensional smooth manifold without boundary:

$$\begin{aligned} \Omega &= \Omega(\mathcal{M}) \quad \text{exterior forms} \\ d &= \text{ the exterior derivative} \\ \int &= \text{ integrals of } k\text{-forms over } \mathcal{M} \end{aligned}$$

This is an n-cycle over $\mathcal{B}' = C^\infty(\mathcal{M})$. Let us check the continuity condition.

$$\forall\, f_1, \ldots, f_n \in \mathcal{B}' = C^\infty(\mathcal{M})\,, \quad \forall\, X_1, \ldots, X_n \in C^\infty(\mathcal{M})$$

$$\left| \int (X_1 df_1) \cdots (X_n df_n) \right| \leq \|X_1\|_\infty \cdots \|X_n\|_\infty \int \underbrace{|df_1 \cdots df_n|}_{\text{this is not } d \text{ of something}}$$

$$\int |df_1 \cdots df_n| =: C(f_1, \ldots, f_n) \text{ is our constant.}$$

Ex 3): Specialize to $\mathcal{M} = \mathrm{S}^1 \rightsquigarrow$ 1-cycle $(\Omega(\mathrm{S}^1), d, \int)$. The character:

$$\begin{aligned} C^\infty(\mathrm{S}^1) \times C^\infty(\mathrm{S}^1) &\to \mathbb{C} \\ (f, g) &\mapsto \int f dg \quad \text{1-trace} \end{aligned}$$

In particular, if f is invertible ($f^{-1} \in C^\infty(\mathrm{S}^1)$ exists $\Leftrightarrow f(x) \neq 0\ \forall\, x \in \mathrm{S}^1$), then

$$(f^{-1}, f) \mapsto \int f^{-1} df$$

is the winding number of f (a factor of 2π included in the measure: $\mathrm{S}^1 \cong \mathbb{R}/\mathbb{Z}$).

What is the winding number? For $f :\to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ it is how many times f wraps around $0 \in \mathbb{C}$. We have defined winding number for a C^∞ function f. One can extend the definition of winding number to any continuous function $f : \mathrm{S}^1 \to \mathbb{C}^*$. One way to think of this is invariance under homotopy:

"Elements of proof." If f is homotopic to g inside $\{\tilde{f} : \mathrm{S}^1 \to \mathbb{C}^* \mid \tilde{f} \in C^\infty(\mathrm{S}^1)\}$ then they have the same winding number. Extend to $\{f : \mathrm{S}^1 \to \mathbb{C}^* \mid f \in C(\mathrm{S}^1)\}$ by saying the winding number of f is the winding number of a smooth function $\tilde{f}$ homotopic to it.

<u>Construction of n-traces</u>. Let $\mathcal{B}$ be a Banach algebra with $\mathbb{R}^n$-action α such that for generators $e_1, \ldots, e_n$ of $\mathbb{R}^n$ the derivation

$$\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \alpha_{te_i}(A) =: \nabla_i A\,, \quad i = 1, \ldots, n$$

exists on a dense domain of $A \in \mathcal{B}$.

Let τ be a (possibly unbounded) positive trace on $\mathcal{B}$ which is α invariant; that is, $\tau(\nabla_i A) = 0$ on an appropriate domain. Suppose

$$\mathcal{B}' = \{A \in \bigcap_j \text{domain} \nabla_j \mid \exists_i : \nabla_i A \text{ is trace class}\}$$

is dense in $\mathcal{B}$.

This yields an n-cycle (unbounded) over $\mathcal{B}$:

Ω: The graded algebra is $\Omega = \mathcal{B} \otimes \bigwedge \mathbb{C}^n$

- $\bigwedge \mathbb{C}^n$ is the Grassmann algebra (= exterior algebra). If we choose a basis $e_1, \ldots, e_n \in \mathbb{C}^n$ then $\bigwedge^k \mathbb{C}^n$ = the linear combinations of $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $i_1, \ldots, i_k \in \{1, \ldots, n\}$, $e_i \wedge e_k = -e_k \wedge e_i$.
- Compare to the exterior algebra $\Omega(\mathcal{M})$ where $\mathcal{M} = \mathbb{T}^n$:

$$\Omega(\mathbb{T}^n) = C^\infty(\mathbb{T}^n) \otimes \underbrace{\bigwedge \mathbb{R}^n}_{\text{constant forms over } \mathbb{T}^n}$$

d: The derivation $d : \Omega^k \to \Omega^{k+1}$ is

$$d(A \otimes v) = \sum_{j=1}^{n} (\nabla_j A) \otimes (e_j \wedge v)\,, \quad A \in \mathcal{B}\,,\, v \in \bigwedge^k \mathbb{C}^n$$

Compare the exterior derivative on $\Omega(\mathcal{M})$

$$\begin{aligned} \omega &= \sum f_{i_1 \cdots i_k} \,\mathrm{d}\, x_{i_1} \wedge \cdots \wedge \mathrm{d}\, x_{i_k} \\ \mathrm{d}\, \omega &= \sum \mathrm{d}\, f_{i_1 \cdots i_k} \,\mathrm{d}\, x_{i_1} \wedge \cdots \wedge \mathrm{d}\, x_{i_k}\,, \text{ where} \\ \mathrm{d}\, f_{i_1 \cdots i_k} &= \sum_{j=1}^{n} \frac{\partial f_{i_1 \cdots i_k}}{\partial x_j} \,\mathrm{d}\, x_j \end{aligned}$$

$\int$: The trace is $\int = \tau \otimes \imath$, with $\imath : \underbrace{\bigwedge^n \mathbb{C}^n}_{\cong \mathbb{C}} \to \mathbb{C}$ given by $\imath(e_1 \wedge e_2 \wedge \cdots e_n) = 1$. I.e., the character has the form

$$\begin{aligned} (A_0, \ldots, A_n) \mapsto \int A_0 \underbrace{dA_1}_{=\sum_j \nabla_j A_1 \otimes e_j} \cdots dA_n & \\ = \sum_{j_1, \ldots, j_n} \tau\left(A_0 (\nabla_{j_1} A_1) \cdots (\nabla_{j_n} A_n)\right) \otimes \imath(e_{j_1} \wedge \cdots \wedge e_{j_n}) & \\ = \sum_{\underbrace{\sigma \in P_n}_{\text{permutations of } n \text{ symbols}}} \mathrm{signum}(\sigma) \tau\left(A_0 (\nabla_{\sigma(n)} A_1) \cdots (\nabla_{\sigma(n)} A_n)\right) & \end{aligned}$$

21 (11 March 2009)

Connes-Thom isomorphism, Chern classes, Bott periodicity

Rappel. An action of $\mathbb{R}^n$ on a Banach algebra $\mathcal{B}$ which is differentiable on a dense sub-algebra gives rise to an n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$.

Aside: Action α of $\mathbb{R}$ on a Banach algebra $\mathcal{B}$ then gives rise to

1) $\mathcal{B} \rightsquigarrow \mathcal{B} \rtimes_\alpha \mathbb{R}$

2) A group homomorphism $K_i(\mathcal{B}) \to K_{i+1}(\mathcal{B} \rtimes_\alpha \mathbb{R})$; in fact the *Connes-Thom isomorphism.*

3) A map: n-cycles over $\mathcal{B} \to (n+1)$-cycles over $\mathcal{B} \rtimes_\alpha \mathbb{R}$.

   Note. $\mathbb{R}^n$-action $\alpha \cong n$-commuting $\mathbb{R}$-actions $\alpha_1, \ldots, \alpha_n$:

$$\begin{aligned} &\alpha_1 : \mathcal{B} \to \mathcal{B} \rtimes_\alpha \mathbb{R} \quad f : \mathbb{R} \to \mathcal{B}\text{, i.e., } \in L^1(\mathbb{R}, \mathcal{B}) \\ \text{then } &\tilde{\alpha}_2(f)(\xi) = \alpha_2(f(\xi)) \\ &\tilde{\alpha}_2 : \mathcal{B} \rtimes_{\alpha_1} \mathbb{R} \to (\mathcal{B} \rtimes_{\alpha_1} \mathbb{R}) \rtimes_{\tilde{\alpha}_2} \mathbb{R} \cong \mathcal{B} \rtimes_{(\alpha_1, \alpha_2)} \mathbb{R}^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(actions commute)} \end{aligned}$$

4) Duality: For any (continuous) α-action of $\mathbb{R}$ on $\mathcal{B}$ there is a dual action $\hat{\alpha}$ of $\hat{\mathbb{R}} = \mathbb{R}$ on $\mathcal{B} \rtimes_\alpha \mathbb{R}$, on the dense set of elements $f \in L^1(\mathbb{R}, \mathcal{B})$, $(\hat{\alpha}_\xi(f))(x) = \mathrm{e}^{i\xi x} f(x)$, a.e. x. (The notation seems strange because $\hat{\alpha}$ does not depend on α.)

Now we can construct $(\mathcal{B} \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$.

Theorem (Takai)**.**

$$\begin{aligned} (\mathcal{B} \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R} &\cong \mathcal{B} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\hat{\alpha}} \mathbb{R} \\ &\cong \mathcal{B} \otimes \underbrace{\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\hat{\alpha}} \mathbb{R}}_{\substack{\cong C_0(\mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}\text{, since} \\ \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \text{ is } C_0(\mathbb{R}) \text{ by FT}}} \\ &\cong \mathcal{B} \otimes \underset{\substack{\text{compact} \\ \text{operators}}}{\mathcal{K}(L^2(\mathbb{R}))} \\ &\underset{\text{SME}}{\sim} \mathcal{B} \end{aligned}$$

(SME = Strong Morita Equivalence)

$\hat{\alpha}$ yields the inverse of the Connes-Thom isomorphism.

SME: compact operators do nothing; however, not so accurate in physics.

Ex: 1) (Starting in 3) with $n=0$.) $\mathcal{A}$ is a C*-algebra, $\mathcal{B} = \underset{\text{suspension}}{\mathrm{S}\,\mathcal{A}} := C_0(\mathbb{R},\mathcal{A}) = C_0(\mathbb{R}\otimes\mathcal{A})$.

Let α be the action on $\mathbb{R}$ by translation. Let τ be a trace on $\mathcal{A}$; τ extends to an α-invariant trace τ^{s} on $C_0(\mathbb{R},\mathcal{A})$:

$$\tau^{\mathrm{s}}(f) = \int_{\mathbb{R}} \tau(f(s))\,\mathrm{d}\,s$$

Note that $\mathrm{S}\,\mathcal{A} \cong \mathcal{A} \rtimes_{\mathrm{id}} \mathbb{R}$ under the FT isomorphism $C_0(\mathbb{R}) \cong \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R}$, and τ^{s} becomes $\tau \circ \mathrm{ev}_0$.

The convolution product on $L^1(\mathbb{R},\mathbb{C})$ = crossed (product) of trivial action of $\mathbb{R}$ on $C_0(\mathbb{R})$. (Recall:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-i\xi x} f(x)\,\mathrm{d}\,x\,;\; f:\mathbb{R}\to\mathcal{A}\,,\; \hat{f}:\mathbb{R}\to\mathcal{A}$$

maps the pointwise product to the convolution product.)

Application of the construction to $\mathcal{B} = \mathrm{S}\,\mathcal{A}$, derivation d given by derivative of the suspension variable on τ^{s}, one gets

$$\Omega = \mathrm{S}\,\mathcal{A}\otimes \textstyle\bigwedge\mathbb{C}\,, \qquad d(f\otimes v) = \underset{\substack{f'=\text{deriv.}\\ e=\text{ one gen.}}}{f'\otimes e\wedge v}\,, \qquad \int = \tau^{\mathrm{s}}\circ\imath$$

$$\imath : \textstyle\bigwedge^1\mathbb{C} \underset{\cong}{\to} \mathbb{C}$$

Do it twice: $\mathcal{B} = \mathrm{S}\,\mathrm{S}\,\mathcal{A} = C_0(\mathbb{R}^2,\mathcal{A})$

$\mathcal{A} = \mathsf{M}_n(\mathbb{C})$, $\tau = \mathrm{Tr}$, then the character of the 2-cycle:

$$(\mathrm{S}\,\mathrm{S}\,\mathcal{A})^3 \ni (f_0,f_1,f_2) \mapsto \frac{1}{2\pi i}\int_{\mathbb{R}^2} \mathrm{Tr}\left(f_0\left(\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right)\right)\mathrm{d}\,x_1\,\mathrm{d}\,x_2$$

What's the 2-cocycle?:

$$\Omega = \mathrm{S}\,\mathrm{S}\,\mathcal{A}\otimes\textstyle\bigwedge\mathbb{C}^2, \quad d(f\otimes v) = \displaystyle\sum_{j=1}^{2}\frac{\partial f}{\partial x_j}\otimes e_j\wedge v, \quad \int = \int_{\mathbb{R}^2}\mathrm{d}\,x_1\,\mathrm{d}\,x_2\circ\tau\circ\imath$$

$$f\in\mathrm{S}\,\mathrm{S}\,\mathcal{A},\; v\in\textstyle\bigwedge\mathbb{C}^2, \qquad \imath:\textstyle\bigwedge^2\mathbb{C}^2\to\mathbb{C}$$

$$e_1\wedge e_2 = 1$$

So,

$$(f_0,f_1,f_2)\mapsto \frac{1}{2\pi i}\int_{\mathbb{R}^2}\mathrm{Tr}(f_0 df_1 df_2) = \frac{1}{2\pi i}\int_{\mathrm{S}^2}\mathrm{Tr}(f_0 df_1 df_2);\ \text{using}$$

$$\mathrm{S}\,\mathrm{S}\,\mathcal{A}^{\,+} = \overset{\text{adjoin a unit}}{C_0(\mathbb{R}^2,\mathcal{A})^+} = C_0(\mathbb{R}^2)^+\otimes\mathcal{A} \cong C(\mathrm{S}^2)\otimes\mathcal{A}$$

Remark. In general, if η is a cyclic cocycle on $\mathcal{B}$ then one can extend η to a cyclic cocycle on $\mathcal{B}^+$. Recall $\mathcal{B}^+ = \mathcal{B} \times \mathbb{C}$ (as a vector space: $\mathcal{B} \oplus \mathbb{C}$) with product

$$(A, \lambda)(A', \lambda') = AA' + A\lambda' + A'\lambda + \lambda\lambda'$$

so that $(0, 1)$ is a unit element in $\mathcal{B}^+$. The extension is

$$\eta^+((A_0, \lambda_0), \ldots, (A_n, \lambda_n)) = \eta(A_0, \ldots, A_n)$$

If $f_0 = f_1 = f_2 = p$, a projection on $\mathrm{S}\,\mathrm{S}\,\mathcal{A}^+ = C(\mathrm{S}^2, \mathsf{M}_n(\mathbb{C}))$; so $p \cong$ VB over S^2. Then

$$\frac{1}{2\pi i} \int_{\mathrm{S}^2} \mathrm{Tr}(pdpdp) = \text{Chern number of the VB}$$

At the heart of Bott periodicity.

22 (23 March 2009)

Recall: If we have an action α of $\mathbb{R}^n$ on a Banach algebra $\mathcal{B}$ such that the derivation

$$\nabla_j = \frac{\mathrm{d}}{\mathrm{d}\,t}\bigg|_{t=0} \alpha_{te_j}\,, \quad j = 1, \ldots, n$$

has dense domain and [we have] an α-invariant trace τ on $\mathcal{B}$ such that

$$\mathcal{B}' = \{A \in \bigcap_{j=1}^{n} \mathrm{dom}(\nabla_j) \,|\, \exists\, j\, : \nabla_j A \text{ is traceclass}\}$$

is dense, then we can construct an n-cycle over $\mathcal{B}$.

Generalization (a trace τ is a 0-cycle; replace τ by an arbitrary k-cycle): Suppose we have a k-cycle $(\Omega, d, \int)$ over $\mathcal{B}$ which is α-invariant; that is, α acts on Ω and i) $\int \circ \alpha = \int$, and ii) α commutes with d. Suppose also that the character of $(\Omega, d, \int)$ is fully defined on $\mathcal{B}'$. Then we can construct a k^{th}-cycle $(\Omega', d', \int')$ on $\mathcal{B}$:

$$\Omega' = \Omega \,\hat{\otimes} \bigwedge \mathbb{C}^n, \text{ where } \hat{\otimes} \text{ is the graded tensor product;}$$

$$d' = d \,\hat{\otimes}\, 1 + \delta, \text{ where } \delta(\omega \,\hat{\otimes}\, v) = \sum_{j=1}^{n} (-1)^j \nabla_j(\omega \,\hat{\otimes}\, e_j \wedge v)\,;$$

$$\textstyle\int' = \int \hat{\otimes}\, \imath, \text{ where } \imath : \bigwedge^n \mathbb{C}^n \to \mathbb{C};\; e_1 \wedge \cdots \wedge e_n \mapsto 1$$

Exercise. If $k = 0$, so that $(\Omega, d, \int) = (\mathcal{B}, 0, \tau)$, then we are getting back the old construction.

Example for crossed products. $\mathcal{B}$ is a C*-algebra with action α of $\mathbb{R}^n$, and τ is an α-invariant trace.

1) Suppose that $\nabla_j = \mathrm{d}(\alpha_{te_j})/\,\mathrm{d}\,t\,|_{t=0}$ has dense domain ($\mathcal{B}'$, as before, is dense). Then we get an n-cycle $(\Omega, d, \int)$ as before.

Task: extend $(\Omega, d, \int)$ to an n-cycle over $\mathcal{B} \rtimes_\alpha \mathbb{R}^n$:

$$\tilde{\Omega} = \Omega \rtimes_\alpha \mathbb{R}^n = (\mathcal{B} \rtimes_\alpha \mathbb{R}^n \,\hat{\otimes} \bigwedge \mathbb{C}^n)$$

$$(\tilde{d}f)(x) = d(f(x)) \text{ if } f \in \Omega \rtimes_\alpha \mathbb{R}^n \text{ (on a dense subset of } f : \mathbb{R}^n \to \Omega)$$

$$\tilde{\textstyle\int} = \textstyle\int \circ\, \mathrm{ev}_0$$

Ex. $\mathcal{B} = C_{\mathcal{P}}(\mathbb{R}^n)$ where $\mathcal{P}$ is a pattern, and α is translation action. This $\rightsquigarrow$ n-trace (n-cycle) over $C_{\mathcal{P}}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n$.

2) *Momentum space instead of configuration space:* construct first the crossed product $\mathcal{B} \rtimes_\alpha \mathbb{R}^n$. On $\mathcal{B} \rtimes_\alpha \mathbb{R}^n$ we have the dual action: $\underset{\xi\in}{\widehat{\mathbb{R}^n}}$ acts via $\tilde{\alpha}$ on $f : \mathbb{R}^n \to \mathcal{B}$

$$(\hat{\alpha}_\xi(f))(x) = \mathrm{e}^{i\xi\cdot x} f(x)$$

We construct an n-cycle according to the recipe (τ is an α invariant trace):

$$\Omega = \mathcal{B} \rtimes_\alpha \mathbb{R}^n \hat{\otimes} \bigwedge \mathbb{C}^n \ (= \tilde{\Omega}\ !)$$

$$\int = (\tau \circ \mathrm{ev}_0) \otimes \imath \ (= \tilde{\int}\ !)$$

$$d(\omega \otimes v) = \sum_{j=1}^{n} (-1)^j \nabla_j \omega \otimes e_j \wedge v$$

but the differential d is different!

$$(\nabla_j f)(x) = i x_j f(x) \ \ \text{(cf. FT)}$$

In our example, $\hat{\alpha}$ is translation in the momentum variables.

"QHE" Rappel: conductivity tensor: linear response to $i\mathcal{L}_{\mathrm{per}} = i\mathcal{L} + \vec{E}\cdot\vec{\delta}$, where $\delta_j = i[\,\hat{q}_j\,, \cdot\,]$

$\underline{n = 2}$.

$$\sigma_{ij} = \frac{1}{2\pi i} \underset{\substack{\uparrow \\ \text{projection onto states of energy} \le \text{Fermi energy} \in \mathrm{Gap}\,(\sigma(H))}}{\tau(P_F}(\delta_i P_F \delta_j P_F - \delta_j P_F \delta_i P_F))\,;$$

$$\sigma_{11} = \sigma_{22} = 0\,;\ \sigma_{12} = -\sigma_{21}\,;$$

$$\mathcal{B} = C_{\mathcal{P}}(\mathbb{R}^n) \cong C(\underset{\substack{\uparrow \\ \text{hull}}}{\Omega_{\mathcal{P}}})$$

$$\forall \omega \in \Omega_{\mathcal{P}}\,,\ (\pi_\omega(f)\psi)(x) = \int \underset{\substack{\uparrow \\ \text{just look at functions of momentum } f:\mathbb{R}^n\to\mathbb{C}\,\in C_{\mathcal{P}}(\mathbb{R}^n)}}{f}(x-y)\psi(y)\,\mathrm{d}\,y\,,\ \ \psi \in L^2(\mathbb{R}^n)$$

What is $\vec{\delta}$ in this representation?

$$
\begin{aligned}
(\pi_\omega(\hat{\alpha}_\xi f)\psi)(x) &= \int \mathrm{e}^{i(x-y)\cdot\xi} f(x-y)\psi(y)\,\mathrm{d}\,y \\
\text{so, } (\pi_\omega(\nabla_j f)\psi)(x) &= \int i(x_j - y_j) f(x-y)\psi(y)\,\mathrm{d}\,y \\
&= \int i[\hat{q}_j, f](x-y)\psi(y)\,\mathrm{d}\,y \\
&= \int i(\hat{q}_j f)(x-y)\psi(y)\,\mathrm{d}\,y - \int i f(x-y)(\hat{q}_j\psi)(y)\,\mathrm{d}\,y \\
&\overset{!}{=} \int (\hat{q}_j f)(x-y)\psi(y)\,\mathrm{d}\,y
\end{aligned}
$$

Hence the conductivity tensor is equal to the pairing of $[P_F]_0 \in K_0(C_0(\mathbb{R}^2) \rtimes \mathbb{R}^2)$ with the character of the 2-cycle coming from the dual action.

23 (25 March 2009)

Summary and more about K-theory

Resumé: We have discussed n-traces (characters of n-cycles) over Banach algebras with the purpose of defining functionals from $K(\mathcal{B}) \to \mathbb{C}$ (a functional defined by an n-trace is a pairing $\langle(\Omega, d, \int) \,|\, x\rangle$, $x \in K(\mathcal{B})$, referred to as a *Connes pairing*).

Ex. A QM system described by a Hamiltonian H affiliated to an algebra $\mathcal{A}$ at temperature $T \searrow 0$.

1) $(n = 0)$ Gap-labelling. $\forall E \notin \sigma(H)$, $\mathrm{IDS}(E) = \underbrace{\langle(\mathcal{B}, 0, \tau) \,|\, [P_E]_0\rangle}_{\text{such a functional}} = \langle \tau \,|\, [P_E]_0\rangle$

2) $(n = 2)$ σ is the tensor of transport coefficients in 1st order response theory associated with a perturbation $\delta = \delta_1, \ldots, \delta_k$. If $\underset{\text{Fermi energy}}{E_F} \notin \sigma(H)$, then

$$\sigma_{ij} = \langle(\Omega, d, \textstyle\int) \,|\, [P_{E_F}]_0\rangle = \langle \tau, \delta_i, \delta_j \,|\, [P_{E_F}]_0\rangle$$

where $(\Omega, d, \int) =$ 2-cycle constructed for the infinitesimal action of δ_i, δ_j, $i < j$:

$$\begin{aligned}
\Omega &= \mathcal{B} \otimes \textstyle\bigwedge \mathbb{C}^2 \\
d(b \otimes v) &= \delta_i(b) \otimes e_i \wedge v - \delta_j(b) \otimes e_j \wedge v \\
\imath &: \underset{e_1 \wedge e_2 \mapsto 1}{\textstyle\bigwedge^2 \mathbb{C}^2 \to \mathbb{C}} \\
\textstyle\int &= \tau \circ \imath
\end{aligned}$$

We can thus describe topologically quantized transport coefficients by means of K-theory.

Aim: using topology to obtain equations between such quantities.

Q [look at 2) and QCs] $\mathcal{A} = \mathcal{A}_{\mathcal{P}} = \overset{\text{config.}}{C_{\mathcal{P}}(\mathbb{R}^n)} \rtimes_\alpha \overset{\text{momentum}}{\mathbb{R}^n}$, $\mathcal{P} =$ an FLC pattern in $\mathbb{R}^n$.

We saw 2 examples related to the 2 copies of $\mathbb{R}^n$. Are there more?

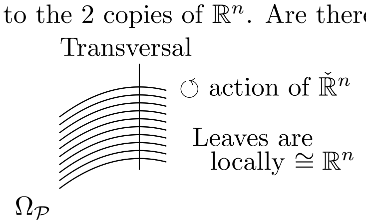


Figure 3: Transversal derivation

Are there reasonable transversal derivations? The transversal is totally disconnected (by FLC).

`(Missing here is the exchange around the question of whether it would suffice to use almost 1-1 surjections onto smooth spaces to create smooth derivations.)`

For QHE $\mathcal{A}_{\mathcal{P}}$ is not sufficient (time reversal $\Rightarrow \sigma$ is symmetric and hence 0, since σ is anti-symmetric by the above).

More about K-theory. K is going to be a functor from the category of (complex, $*$)-Banach algebras to the category of abelian groups satisfying certain functorial properties (which determine K in that it is possible to construct K from these properties). In addition to the functor properties:

$$f : \mathcal{A} \to \mathcal{B} \text{ ($*$-algebra morphism)} \mapsto K(f) : K(\mathcal{A}) \to K(\mathcal{B}) \text{ (group morphism)}$$
$$K(\mathrm{id}) = \mathrm{id}$$
$$K(f \circ g) = K(f) \circ K(g)$$

we want

1) K is $\mathbb{Z}$-*graded*; i.e., $\forall n \in \mathbb{Z}$ we have a functor K_n (satisfying the above functor properties);

2) K is *homological*: from a short exact sequence of Banach algebras

$$0 \to \mathcal{J} \xrightarrow{\imath} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$$

$\imath$ is injective; $\mathrm{Im}\,\imath = \mathrm{Ker}\,q$; q is surjective

we get a long exact sequence (with boundary or connecting maps ∂_n which are group homomorphisms)

$$\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{J}) \xrightarrow{K_n(\imath)} K_n(\mathcal{E}) \xrightarrow{K_n(q)} K_n(\mathcal{A}) \xrightarrow{\partial_n} K_{n-1}(\mathcal{J})$$
$$\xrightarrow{K_{n-1}(\imath)} K_{n-1}(\mathcal{E}) \xrightarrow{K_{n-1}(q)} K_{n-1}(\mathcal{A}) \xrightarrow{\partial_{n-1}} \cdots$$

Im(incoming morphism)=Ker(outgoing morphism) at each group

24 (27 March 2009)

K-theory as a homology functor, cont.

Short exact sequence of C*-algebras:

$$(*) \qquad 0 \to \mathcal{J} \to \mathcal{E} \to \mathcal{B} \to 0$$

If $\mathcal{E}$ is unital and the ideal $\mathcal{J}$ is also unital, then[10] $\mathcal{J} = \mathcal{E}$. Recall that $(*)$ is equivalent to saying that $\mathcal{J}$ is a (closed, two-sided) ideal in $\mathcal{E}$; i. e., $\mathcal{J}$ is a closed sub-algebra of $\mathcal{E}$ which satisfies $je \in \mathcal{J}$ and $ej \in \mathcal{J}$, $\forall\, j \in \mathcal{J}$, $e \in \mathcal{E}$. So we need K for non-unital C*-algebras.

Remark. If $\mathcal{B}$ is separable, commutative and non-unital then $\mathcal{B} \cong C_0(X)$, X non-compact (but locally compact). You all know that compactly supported singular cohomology is complicated.

Definition (of $K_0(\mathcal{B})$)**.**

1) Recall the definition for the unital case.

   i) Introduce an equivalence relation on projections:

$$p \underset{\mathrm{h}}{\sim} q \text{ if } \exists \text{ continuous path } [0,1] \to \mathcal{B} \text{ such that}$$
$$\forall\, t \in [0,1] : p(t)^2 = p(t) = p(t)^*\,, \text{ and, } p(0) = p,\ p(1) = q.$$

   ii) Stabilization:

$$\underbrace{\mathsf{M}_n(\mathcal{B})}_{n\times n \text{ matrices over } \mathcal{B}} \hookrightarrow \mathsf{M}_{n+1}(\mathcal{B})$$

$$x \in \mathsf{M}_n(\mathcal{B}) \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathsf{M}_{n+1}(\mathcal{B})$$

   and then pass to

$$\bigcup_{n\geq 1} \mathsf{M}_n(\mathcal{B}) \text{ or } \varinjlim(\mathsf{M}_n(\mathcal{B}) \hookrightarrow \mathsf{M}_{n+1}(\mathcal{B}))$$

   (the closure of $\lim_{\to}(\mathsf{M}_n(\mathcal{B}) \hookrightarrow \mathsf{M}_{n+1}(\mathcal{B}))$ is $\mathcal{K} \otimes \mathcal{B}$ where $\mathcal{K} = \mathcal{K}(\ell^2)$ = compact operators on some separable Hilbert space).

[10] See the 30 March lecture for more details.

iii) Monoid structure: $\mathcal{V}(\mathcal{B}) = \{$equivalence classes of projections in $\bigcup_{n\geq 1} \mathsf{M}_n(\mathcal{B})\}$, and define $[p]+[q]$, $p \in \mathsf{M}_n(\mathcal{B}) \hookrightarrow \mathsf{M}_{n+m}(\mathcal{B})$, $q \in \mathsf{M}_m(\mathcal{B}) \hookrightarrow \mathsf{M}_{n+m}(\mathcal{B})$, by

$$\begin{aligned} [p]+[q] &= \left[\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right], \text{ using } \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right] \end{aligned}$$

(Works because the sum of two projections is a projection if they are orthogonal.)

Also, $[0]$ is a neutral element, and so $(\mathcal{V}(\mathcal{B}), +)$ is a monoid.

iv) Grothendieck construction applied to $\mathcal{V}(\mathcal{B})$:

$$\begin{gathered} K_0(\mathcal{B}) = \mathcal{V}(\mathcal{B}) \times \mathcal{V}(\mathcal{B}) / \sim \text{ , where} \\ \underbrace{([p],[q])}_{\text{think of } [p]-[q]} \sim ([p'],[q']) \text{ if } \exists\, [r] \in \mathcal{V}(\mathcal{B}) \text{ such that} \\ [p]+[q']+[r] = [p']+[q]+[r] \end{gathered}$$

2) Now suppose that $\mathcal{B}$ is a non-unital C*-algebra. The definition of K_0 given for a unital algebra makes perfect sense here, so call the result $K_{00}(\mathcal{B})$ (which we do not like because it is not homological). K_{00} is a functor: if $f : \mathcal{A} \to \mathcal{B}$ is an algebra morphism, then $K_{00}(f) : K_{00}(\mathcal{A}) \to K_{00}(\mathcal{B})$ is given by

$$[[p],[q]] \mapsto [[f(p)],[f(q)]]$$

v) Unitization (add a unit):

$$\begin{gathered} \mathcal{B}^+ = \mathcal{B} \times \mathbb{C} \text{ as VS, and define a product} \\ (b,\lambda)(b',\lambda') = (bb' + \lambda b' + \lambda' b, \lambda\lambda') \end{gathered}$$

(If $\mathcal{B} = C_0(X^{\text{loc cmpct}})$, then $\mathcal{B}\times\mathbb{C} \cong C_0(X) + \{\lambda : X \to \mathbb{C}, \text{constant}\}$.)

We get a SES (short exact sequence):

$$0 \to \mathcal{B} \underset{b \mapsto (b,0)}{\hookrightarrow} \mathcal{B}^+ \underset{(b,\lambda)\mapsto\lambda}{\xrightarrow{\pi}} \mathbb{C} \to 0$$

vi) Definition. $K_0(\mathcal{B}) = \operatorname{Ker} K_{00}(\pi)$

$$\begin{gathered} K_{00}(\pi) : K_{00}(\mathcal{B}^+) \to K_{00}(\mathbb{C}) \cong \mathbb{Z} \\ \underset{1 \text{ is the added unit } \in \mathcal{B}^+}{[[1],[0]]} \quad \mapsto \quad \underset{\text{the generator}}{[1]} \end{gathered}$$

So $K_0(\mathcal{B})$ is everything "orthogonal" to the element $[[1],[0]]$.

Ex. 1) $\mathcal{B} = C_0(\mathbb{R})$

$$\mathcal{B}^+ = C_0(\mathbb{R})^+ \cong C(\mathrm{S}^1)$$

$K_{00}(C(\mathrm{S}^1)) \cong \mathbb{Z}$ (eq classes of complex VBs over S^1; all trivial), so
$\mathrm{Ker}\, K_{00}(\pi) = 0$, and so
$K_0(C_0(\mathbb{R})) = 0$

Ex. 2) $\mathcal{B} = C_0(\mathbb{R}^2)$, $\mathcal{B}^+ = C(\mathrm{S}^2)$

$K_{00}(C(\mathrm{S}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$; one generator the trivial line bundle $[[1],[0]]$
the other generator = the canonical line bundle $[[b],[0]]$

The generator $[[b],[0]]$ is the Bott-projection:

$$b : \mathbb{R}^2 \to \mathsf{M}_2(\mathbb{C})\,,\ (b \in \mathsf{M}_2(C_0(\mathbb{R}^2)^+)),$$
$$b(x_1,x_2) = \frac{1}{1+x_1^2+x_2^2}\begin{pmatrix} 1 & x_1+ix_2 \\ x_1-ix_2 & x_1^2+x_2^2 \end{pmatrix}$$
$$\lim_{\|(x_1,x_2)\|\to\infty} b(x_1,x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$b^2 = b = b^*$$

So,

$$K_{00}(\pi)[[b],[0]] = [[\pi(b)],[0]] = [[\lim_{\|(x_1,x_2)\|\to\infty} b(x_1,x_2)],[0]]$$
$$= [[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}],[0]] = [[1],[0]]$$

So $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$ generated by $[[b],[1]]$; the difference between two line bundles.

25 (30 March 2009)

K-theory as a homology functor, cont.

Correction. $\mathcal{A}^+ = \mathcal{A} \times \mathbb{C}$ with product $(a, \lambda)(a', \lambda') = (aa' + \lambda a' + \lambda' a, \lambda\lambda')$ has involution $(a, \lambda)^* = (a^*, \bar{\lambda})$, and norm[11]

$$\|(a, \lambda)\| = \max\{|\lambda|, \sup_{a' \in \mathcal{A}\,,\, \|a'\| \leq 1} \|aa' + \lambda a'\|\}$$

We have the following split exact sequence

$$0 \to \mathcal{A} \overset{i_1}{\hookrightarrow} \mathcal{A}^+ \underset{s}{\overset{\mathrm{pr}_2}{\rightleftarrows}} \mathbb{C} \to 0, \;\; \mathrm{pr}_2 \circ s = \mathrm{id}, \;\; \text{where}$$
$$s : \mathbb{C} \to \mathcal{A}^+; \;\; \lambda \mapsto (0, \lambda)\,, (*\text{-algebra homomorphism})$$

Warning: this does not imply that $\mathcal{A}^+ \cong \mathcal{A} \oplus \mathbb{C}$ as $*$-algebras; only as an isomorphism of vector spaces.

Lemma. *Let $0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$ be a SES of C*-algebras. Then $\mathcal{E} \cong \mathcal{J} \oplus \mathcal{A}$ (as C*-algebras) iff $\exists\, \nu : \mathcal{E} \to \mathcal{J}$, $*$-algebra map, such that $\nu \circ i = \mathrm{id}$*

Proof. The isomorphism is given by

$$\begin{aligned} &\mathcal{E} \to \mathcal{J} \oplus \mathcal{A} \\ &x \mapsto (\nu(x), q(x)) \end{aligned}$$

□

If $\mathcal{A}$ is unital ($e \in \mathcal{A}$ the unit), and $\mathcal{A}^+$ is the unitization,

$$0 \to \mathcal{A} \underset{\nu}{\rightleftarrows} \mathcal{A}^+ \to \mathbb{C} \to 0\,, \;\; \nu(a, \lambda) = a + e\lambda$$

Now ν is a $*$-algebra homomorphism. Conclusion: If $\mathcal{A}$ is unital then $\mathcal{A}^+ \cong \mathcal{A} \oplus \mathbb{C}$. Note that the unit of $\mathcal{A}$ is not mapped to the unit of $\mathcal{A}^+$.

Theorem. *K_0 is a covariant functor from Banach algebras $\to$ abelian groups satisfying the following properties*

1) The stuff all cov functors satisfy:

$$K_0(\mathrm{id} : \mathcal{A} \to \mathcal{A}) = \mathrm{id} : K_0(\mathcal{A}) \to K_0(\mathcal{A})$$
$$K_0(\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}) = K_0(\mathcal{A}) \xrightarrow{K_0(f)} K_0(\mathcal{B}) \xrightarrow{K_0(g)} K_0(\mathcal{C})$$

[11] The definitions of the $*$ and norm were omitted in the previous lecture.

$$K_0(0:\mathcal{A}\to\mathcal{B}) = 0 : K_0(\mathcal{A}) \to K_0(\mathcal{B})$$

2) Stability: $K_0(\mathsf{M}_n(\mathcal{A})) \cong K_0(\mathcal{A})$ *with isomorphism induced by*

$$K_0(\mathcal{A} \overset{i_n}{\hookrightarrow} \mathsf{M}_n(\mathcal{A}))$$

with $i_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

Consequently:

$$\underset{\text{compact op's}}{\mathcal{K}(\ell^2)} = \mathcal{K} = \overline{\bigcup_n \mathsf{M}_n(\mathbb{C})}^{C^*} = \lim_{n\to\infty} \mathsf{M}_n(\mathbb{C})$$
$$\Rightarrow K_0(\mathcal{A}\otimes\mathcal{K}) \cong K_0(\mathcal{A})$$

3) Homotopy invariance: $\phi_0 \underset{\mathrm{h}}{\sim} \phi_1 : \mathcal{A} \to \mathcal{B}$ *($*$-algebra maps) if*

$$\exists\,(\phi_t : \mathcal{A}\to\mathcal{B})_{t\in[0,1]} (*\text{-algebra maps) such that}$$
$$\phi_0 = \phi_t|_{t=0}\,,\ \phi_1 = \phi_t|_{t=1} \ \text{and}$$
$$\forall\, a \in \mathcal{A}\,,\ t \mapsto \phi_t(a) \ \text{is continuous}$$

i) If $\phi_0 \underset{\mathrm{h}}{\sim} \phi_1 : \mathcal{A}\to\mathcal{B}$, *then* $K_0(\phi_0) = K_0(\phi_1)$

ii) If $\mathcal{A} \underset{\mathrm{h}}{\sim} \mathcal{B}$, *that is,* $\exists\, *$*-alg maps* $\mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{A}$ *such that* $\phi\circ\psi \underset{\mathrm{h}}{\sim} \mathrm{id}_{\mathcal{B}}$ *and* $\psi\circ\phi \underset{\mathrm{h}}{\sim} \mathrm{id}_{\mathcal{A}}$, *then* $K_0(\phi)$ *provides an isomorphism between* $K_0(\mathcal{A})$ *and* $K_0(\mathcal{B})$.

Application. $\mathcal{A}$ any algebra; $\underset{\text{cone}}{\mathrm{C}}\,\mathcal{A} = C_0((0,1],\mathcal{A})$

$$C_0((0,1],\mathcal{A}) = \{f:(0,1]\to\mathcal{A}\ (\text{cont}) \mid \lim_{t\to 0} f(t) = 0\}$$
$$= \{f:[0,1]\to\mathcal{A}\ (\text{cont}) \mid f(0) = 0\}$$

Then $\mathrm{C}\,\mathcal{A} \underset{\mathrm{h}}{\sim} \{0\}$ as follows:

$$s\in[0,1]\,,\ \tilde{\phi}_s : \mathrm{C}\,\mathcal{A}\to\mathrm{C}\,\mathcal{A}\,,\ (\tilde{\phi}_s(f))(t) := f(st)$$
$$\tilde{\phi}_0 = 0 : \mathrm{C}\,\mathcal{A}\to\mathrm{C}\,\mathcal{A}\,,\ \text{and } \tilde{\phi}_1 = \mathrm{id} : \mathrm{C}\,\mathcal{A}\to\mathrm{C}\,\mathcal{A}$$
$$\Rightarrow \text{for } \mathrm{C}\,\mathcal{A} \xrightarrow{\phi} 0 \xrightarrow{\psi} \mathrm{C}\,\mathcal{A}\,,\ \text{we have}$$
$$\phi\circ\psi = 0 = \mathrm{id} : 0 \to 0\,,\ \text{and } \psi\circ\phi = 0 \underset{\mathrm{h}}{\sim} \mathrm{id} : \mathrm{C}\,\mathcal{A}\to\mathrm{C}\,\mathcal{A}$$

The cone is contractible. Hence $K_0(\mathrm{C}\,\mathcal{A}) = 0$.

4) Half-exactness: *If* $0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$ *is a SES of Banach $*$-algebras, then* $K_0(\mathcal{J}) \xrightarrow{K_0(i)} K_0(\mathcal{E}) \xrightarrow{K_0(q)} K_0(\mathcal{A})$ *is exact:* $\mathrm{Im}\,K_0(i) = \mathrm{Ker}\,K_0(q)$.

5) Split SESs and direct sums: *If*

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \underset{s}{\overset{q}{\rightleftarrows}} \mathcal{A} \to 0$$

is a split SES, then

$$0 \to K_0(\mathcal{J}) \xrightarrow{K_0(i)} K_0(\mathcal{E}) \underset{K_0(s)}{\overset{K_0(q)}{\rightleftarrows}} K_0(\mathcal{A}) \to 0$$

is a split SES. In particular, $K_0(\mathcal{E}) \cong K_0(\mathcal{J}) \oplus K_0(\mathcal{A})$.

Application: $K_0(\mathcal{A}^+) \cong K_0(\mathcal{A}) \oplus \underbrace{\mathbb{Z}}_{=K_0(\mathbb{C})}$.

Remarque: $K_0(\mathcal{A}) = \mathrm{Ker}(K_{00}(\pi))$, $\mathcal{A}^+ = \mathcal{A} \times \mathbb{C} \xrightarrow{\pi} \mathbb{C}$. The verification is but tedious once one realizes that if $\mathcal{A} \xrightarrow{f} \mathcal{B}$ is a morphism, then $\mathcal{A}^+ \xrightarrow{f^+} \mathcal{B}^+$ with

$$f^+(a, \lambda) = (f(a), \lambda)$$

is a canonical extension.

Lemma. *Any element in* $\mathrm{Ker}(K_{00}(\pi))$ *can be written as follows:*

$$[[p], [1_n]] \quad (\textit{notation } [p]_0 - [1_n]_0)$$

where $[p]$ *is the eq class of a projection in* $\mathsf{M}_n(\mathcal{A})$*, and* $[1_n]$ *is the eq class of*

$$\underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}}_{n \times n} \quad (1 \textit{ is the added unit in } \mathcal{A}^+)$$

26 (1 April 2009)

K-theory as a homology functor, cont.

Recall that if $0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$ is a SES of $*$-Banach algebras then $K_0(\mathcal{J}) \xrightarrow{K_0(i)} K_0(\mathcal{E}) \xrightarrow{K_0(q)} K_0(\mathcal{A})$ is exact (one says that K_0 is *half-exact*; it is also *split-exact*).

But $K_0(i)$ is not necessarily injective, and $K_0(q)$ is not necessarily surjective.

The first case ($K_0(i)$ is not injective):

$$0 \to \underset{\substack{\text{compact}\\\text{operators}}}{\mathcal{K}(\mathcal{H})} \xrightarrow{i} \mathcal{B}(\mathcal{H}) \xrightarrow{q} \underset{\text{Calkin algebra}}{\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})} \to 0$$

($\mathcal{H}$ is a separable, infinite dimensional Hilbert space; $\mathcal{H} = \ell^2$),

By stability, $K_0(\mathcal{K}(\mathcal{H})) \cong K_0(\mathbb{C}) = \mathbb{Z}$.

For $K_0(\mathcal{B}(\mathcal{H}))$, since $\dim \mathcal{H} = \infty$, $\mathsf{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n}) \cong \mathcal{B}(\mathcal{H})$ (all infinite dimensional, separable Hilbert spaces are unitarily equivalent!), so the question boils down to looking at projection classes in $\mathcal{B}(\mathcal{H})$.

$p, q \in \mathcal{B}(\mathcal{H})$ (two projections).

i) $\dim q = \dim \operatorname{Im} q = \operatorname{Tr}(q) < \infty$.
If $p \underset{\text{h}}{\sim} q$, then $\operatorname{Tr}(p) = \operatorname{Tr}(q)$ because Tr is a homotopy invariant. (The converse is also valid: $\operatorname{Tr}(p) = \operatorname{Tr}(q) < \infty \Rightarrow p \underset{\text{h}}{\sim} q$.) So the homotopy classes of finite dimensional projections are uniquely classified by the dimension (trace).

ii) $\dim q = \infty \Leftrightarrow q(\mathcal{H})$ is an ∞-dimensional Hilbert space $\Leftrightarrow q(\mathcal{H}) \underset{\substack{\mathsf{M}\\\text{unitary}}}{\cong} \mathcal{H} = 1(\mathcal{H}) \Rightarrow q \underset{\text{h}}{\sim} 1$.
All infinite dimensional projections are homotopic!

Now $\mathcal{V}(\mathcal{B}(\mathcal{H})) \cong \mathbb{N}_0 \cup \{+\infty\} \ni n, m, \quad n + m \cong$ adding projections, and

$$\operatorname{Tr}\begin{pmatrix} p & \\ & q \end{pmatrix} = \operatorname{Tr}(p) + \operatorname{Tr}(q)$$

This is the monoid. What is the Grothendieck group?

$$\begin{gathered} K_0(\mathcal{B}(\mathcal{H})) = (\mathbb{N}_0 \cup \{+\infty\}) \times (\mathbb{N}_0 \cup \{+\infty\}) / \sim, \text{ where} \\ ([p],[q]) \sim ([p'],[q']) \text{ if } \exists\, [r] \text{ such that} \\ [p] + [q'] + [r] = [p'] + [q] + [r] \end{gathered}$$

These are now numbers:

$$(n, m) \sim (n', m') \text{ if } \exists\, r \text{ such that } n + m' + r = n' + m + r$$

This is always satisfied if $r = +\infty$! So all elements are equivalent! $K_0(\mathcal{B}(\mathcal{H})) = \{0\}$. So $K_0(i)$ is not injective:

$$K_0(\mathcal{K}(\mathcal{H})) = \mathbb{Z} \overset{K_0(i)}{\to} \{0\} = K_0(\mathcal{B}(\mathcal{H}))$$

Re the Calkin algebra we have *Atkinson's theorem* which says that an operator in $\mathcal{B}(\mathcal{H})$ is Fredholm $\Leftrightarrow$ its image in the Calkin algebra is invertible.

<u>Other Ex.</u> ($K_0(q)$ is not surjective): $\mathcal{A}$ any C*-algebra (or $*$-Banach algebra)

$$\text{SES:} \quad 0 \to \underset{\substack{\text{suspension} \\ C_0((0,1),\mathcal{A})}}{\mathrm{S}\,\mathcal{A}} \xrightarrow{i} \underset{\substack{\text{cone} \\ C_0((0,1],\mathcal{A})}}{\mathsf{C}\mathcal{A}} \xrightarrow{\mathrm{ev}_1} \mathcal{A} \to 0$$

$$\text{(note: } C_0((0,1),\mathcal{A}) = \{f : [0,1] \underset{\text{cont}}{\to} \mathcal{A} \mid f(0) = f(1) = 0\}\text{)}$$

This gives

$$K_0(\mathrm{S}\,\mathcal{A}) \xrightarrow{K_0(i)} \underset{\substack{\mathsf{C}\mathcal{A}\text{ is} \\ \text{contractible}}}{\{0\}} \xrightarrow{K_0(\mathrm{ev}_1)} K_0(\mathcal{A})$$

In particular, $K_0(\mathrm{ev}_1)$ is not necessarily onto (e.g., $\mathcal{A} = \mathbb{C}$)

Compactly supported singular cohomology.

Relative cohomology groups $H^*(X, A)$, $A \subset X$ topological spaces.

Singular chains $C_n(A) \subset C_n(X)$.

$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \longrightarrow C_n(X)/C_n(A) \longrightarrow 0$$

This gives LES of homology and cohomology groups where by definition $H(X, A)$ is the homology or cohomology of the quotient.

Definition (Compactly supported singular cohomology)**.**

$$H_c^*(X) = \lim_{K \nearrow X} H^*(X, X \setminus K)$$

the limit taken over compact sets K exhausting all of X.

Compactify X by one point: $X^+ = X \cup \{\infty\} = \mathrm{spec}(C_0(X)^+)$. Apply excision[12] $\{\infty\} \subset U = X^+ \setminus K \subset X^+ \Rightarrow H^*(X^+, X^+ \setminus K) \cong H^*(X, X \setminus K)$. Then

$$H_c^*(X) = \lim_{U \searrow \infty} H^*(X^+, U)$$

[12] $\{\infty\}$ is a closed subset of the open set $X^+ \setminus K$

the limit taken over open sets U shrinking down to the point ∞.

LES in cohomology of the pair (X^+, U)

$$0 \leftarrow H^0(U) \leftarrow H^0(X^+) \leftarrow H^0(X^+, U) \leftarrow \cdots$$
$$\cdots \leftarrow H^n(U) \leftarrow H^n(X^+) \leftarrow H^n(X^+, U) \leftarrow \cdots$$

Now

$$\lim_{U \searrow \{\infty\}} H^n(X^+, U) = H^n_c(X)$$

and for spaces with nbhds U of which ∞ is a retract; e.g., $\mathbb{R}^n$,

$$\lim_{U \searrow \{\infty\}} H^n(U) = H^n(\{\infty\})$$

so

$$0 \leftarrow \underbrace{H^0(\mathrm{pt})}_{\cong K_0(\mathbb{C})} \leftarrow H^0(X^+) \leftarrow H^0_c(X) \leftarrow 0, \text{ and}$$
$$H^n(X^+) \cong H^n_c(X) \text{ for } n > 0$$

27 (3 April 2009)

Introducing $K_1(\mathcal{A})$

$$\begin{aligned}\mathcal{A} &= \text{a C*-algebra with unit;}\\ \mathcal{U}(\mathcal{A}) &= \{\text{unitary elements of } \mathcal{A}\}\\ &= \{u \in \mathcal{A} \,|\, u \text{ is invertible and } u^{-1} = u^*\};\\ u \in \mathcal{U}(\mathcal{A}) \Rightarrow \sigma(u) &\subset \mathbb{T}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}\end{aligned}$$

Lemma. *If $u \in \mathcal{U}(\mathcal{A})$ and $\sigma(u) \neq \mathbb{T}^1$, then $u \underset{\mathrm{h}}{\sim} 1$ by a homotopy in $\mathcal{U}(\mathcal{A})$.*

Proof. The important point is that $\ln_\theta(u) \in \mathcal{A}$ (see Fig. 4).

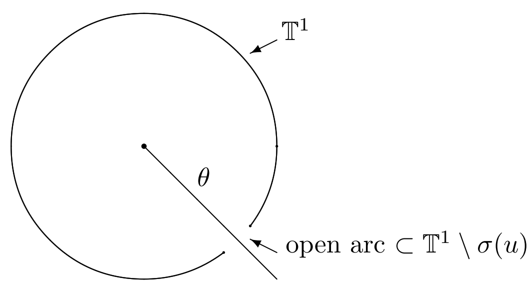


Figure 4: Cut θ defining $\ln_\theta(u)$

Now let $u_t = \mathrm{e}^{it\ln_\theta(u)}$. Then u_t is unitary since $\ln_\theta(u)$ is self-adjoint (real). And $t \in [0,1] \mapsto u_t \in \mathcal{U}(\mathcal{A})$ is continuous, with $u_0 = 1$, and $u_1 = u$. $\square$

Lemma. *For any separable Hilbert space $\mathcal{H}$, $\mathcal{U}(\mathcal{B}(\mathcal{H})) = \mathcal{U}(\mathcal{B}(\mathcal{H}))_0$, where $\mathcal{U}(\mathcal{B}(\mathcal{H}))_0$ is the connected component of 1.*

Proof. If $\dim \mathcal{H} < \infty$, then each unitary (and each operator) has discrete spectrum and the proof of the previous lemma carries over.

In the general case $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra[13] and is closed under the functional calculus for Borel measurable functions: if $f : \mathbb{C} \to \mathbb{C}$ is Borel measurable, then $f(A) \in \mathcal{B}(\mathcal{H}), \forall A \in \mathcal{B}(\mathcal{H})$. Since a cut has measure 0, $\ln : \mathbb{C} \to \mathbb{C}$ is Borel measurable;[14] so $\ln(u) \in \mathcal{B}(\mathcal{H})$. The same argument as before, $u_t = \mathrm{e}^{it\ln(u)}$, yields a homotopy between 1 and u. $\square$

[13] The algebra $\mathcal{B}(\mathcal{H})$ is closed under ??? for separable?? Hilbert space $\mathcal{H}$. The algebra $\mathcal{B}(\mathcal{H})$ is not separable when $\dim \mathcal{H} = \infty$.

[14] A cut in the complement of $\sigma(u)$.

If $\mathcal{A}$ is a unital $*$-Banach algebra:

$$\mathcal{U}(\mathsf{M}_n(\mathcal{A})) \hookrightarrow \mathcal{U}(\mathsf{M}_m(\mathcal{A})) \quad (m > n)$$
$$a \mapsto \underbrace{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}}_{\text{(to be in a group)}}, \quad a \in \mathcal{U}(\mathsf{M}_n(\mathcal{A}))$$

Let

$$\mathcal{U}_\infty(\mathcal{A}) := \bigcup_n \mathcal{U}(\mathsf{M}_n(\mathcal{A})) = \varinjlim_n \mathcal{U}(\mathsf{M}_n(\mathcal{A}))$$

The multiplication in $\mathcal{U}(\mathcal{A})$ induces a multiplication in $\mathcal{U}(\mathcal{A})/\underset{\mathrm{h}}{\sim}$. ($\mathcal{U}(\mathcal{A})_0$ is a normal subgroup.) We do this for $\mathcal{U}_\infty(\mathcal{A})$. So we get a multiplication on $\mathcal{U}_\infty(\mathcal{A})/\underset{\mathrm{h}}{\sim}$.

Lemma (Whitehead lemma)**.** *$\mathcal{U}_\infty(\mathcal{A})/\underset{\mathrm{h}}{\sim}$ is abelian.*

Proof. For $u, v \in \mathcal{U}_\infty(\mathcal{A})$, $[u][v] = [v][u] \Leftrightarrow uv \underset{\mathrm{h}}{\sim} vu$.

So, for $u, v \in \mathsf{M}_n(\mathcal{A})$,

$$\underset{\in \mathsf{M}_n(\mathcal{A})}{uv} = \underset{\in \mathsf{M}_{2n}(\mathcal{A})}{\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

note: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underset{\mathrm{h}}{\sim} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; same spectrum so homotopic

$$\underset{\mathrm{h}}{\sim} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$
$$= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

Similarly

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \underset{\mathrm{h}}{\sim} \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$$

and so

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \underset{\mathrm{h}}{\sim} \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}$$

□

Notation: $[u] + [v] = [uv] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]$ and $[u^{-1}] = -[u]$.

For unital algebras, $K_1(\mathcal{A}) := \mathcal{U}_\infty(\mathcal{A})/\underset{\mathrm{h}}{\sim}$. For non-unital algebras the definition makes no sense.

Definition. Let $\mathcal{B}$ be a $*$-Banach algebra (unital or not).

$$K_1(\mathcal{B}) := \mathcal{U}_\infty(\mathcal{B}^+)/\underset{\mathrm{h}}{\sim}$$

Why does this work? The split SES for unitization:

$$0 \to \mathcal{B} \overset{i}{\hookrightarrow} \mathcal{B}^+ \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0,\ \pi \circ s = \mathrm{id}$$

We saw: $\mathcal{U}_\infty(\mathbb{C})/\underset{\mathrm{h}}{\sim} = 0$, so if taking $\mathcal{U}_\infty(\,\cdot\,)/\underset{\mathrm{h}}{\sim}$ is a split exact functor, then

$$\mathcal{U}_\infty(\mathcal{B})/\underset{\mathrm{h}}{\sim} \cong \mathcal{U}_\infty(\mathcal{B}^+)/\underset{\mathrm{h}}{\sim} \quad \text{for unital } \mathcal{B}$$

Proposition. *K_1 is a functor which satisfies*

1) *If $f : \mathcal{A} \to \mathcal{B}$ is a $*$-Banach algebra morphism, then $f^+ : \mathcal{A}^+ \to \mathcal{B}^+$, $f^+(a, \lambda) = (f(s), \lambda)$, extending to $\mathcal{U}_\infty(\mathcal{A}^+)$ since f^+ is continuous and unital ($f(0,1) = (0,1)$), it preserves homotopy and maps $\mathcal{U}_\infty(\mathcal{A}^+)_0$ to $\mathcal{U}_\infty(\mathcal{B}^+)_0$, so*

$$K_1(f)([u]) = [f(u)]$$
$$K_1(\mathrm{id}) = \mathrm{id}$$
$$K_1(f \circ g) = K_1(f) \circ K_1(g)$$

2) Stability: $K_1(\mathcal{B}) \cong K_1(\mathsf{M}_n(\mathcal{B})) \cong K_1(\mathcal{K} \otimes \mathcal{B})$ *with isomorphism induced by*

$$\mathcal{B} \hookrightarrow \mathsf{M}_n(\mathcal{B}),\ b \mapsto \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

3) Homotopy invariance.

4) Half-exact: *If* $0 \to \mathcal{J} \overset{i}{\to} \mathcal{E} \overset{q}{\to} \mathcal{A} \to 0$ *is a SES of Banach $*$-algebras, then* $K_1(\mathcal{J}) \overset{K_1(i)}{\to} K_1(\mathcal{E}) \overset{K_1(q)}{\to} K_1(\mathcal{A})$ *is exact:* $\mathrm{Im}\, K_1(i) = \mathrm{Ker}\, K_1(q)$.

5) Split exact.

The next goal is to establish boundary maps *index* and *exp*. If

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$$

is a SES of Banach $*$-algebras, then $K_2 \cong K_0$ and

$$\begin{tikzcd}
K_1(\mathcal{J}) \arrow[r, "K_1(i)"] & K_1(\mathcal{E}) \arrow[r, "K_1(q)"] & K_1(\mathcal{A}) \arrow[dll, "\text{index}"] \\
K_0(\mathcal{J}) \arrow[r, "K_0(i)"] & K_0(\mathcal{E}) \arrow[r, "K_0(q)"] & K_0(\mathcal{A}) \arrow[ull, "\exp"]
\end{tikzcd}$$

Figure 5: Boundary maps

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$K_1(\mathcal{A})$ (continued)

Lemma. *If $\mathcal{B}$ is unital (with SES of unitization $0 \to \mathcal{B} \underset{\nu}{\rightleftarrows} \mathcal{B}^+ \to \mathbb{C} \to 0$; $\nu((b,\lambda)) = b + \lambda e$, e the unit in $\mathcal{B}$), then the map ρ defined by the following commuting diagram is an isomorphism:*

$$\begin{array}{ccc} \mathcal{U}_\infty(\mathcal{B}^+) & \xrightarrow{\nu} & \mathcal{U}_\infty(\mathcal{B}) \\ \underset{\mathrm{h}}{\sim}\downarrow & & \downarrow \\ K_1(\mathcal{B}) & \xrightarrow{\rho} & \mathcal{U}_\infty(\mathcal{B})/\underset{\mathrm{h}}{\sim} \end{array}$$

Proof. A good exercise. $\square$

Ex.: $K_1(C_0(\mathbb{R})) = \mathcal{U}_\infty(C_0(\mathbb{R})^+)/\underset{\mathrm{h}}{\sim}$, with

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \overset{\text{conts}}{\longrightarrow} \mathbb{C} \mid \lim_{x\to\pm\infty} f(x) = 0\} \cong C_0(]0,1[) = \underset{\text{suspension}}{\mathrm{S}\,\mathbb{C}}$$

Recall that $C_0(\mathbb{R})^+ \cong C(\mathrm{S}^1)$, so $K_1(\mathrm{S}\,\mathbb{C}) \cong K_1(C(\mathrm{S}^1))$.

Remarque: One can do $\mathcal{U}(\mathsf{M}_n(\mathcal{B}))/\underset{\mathrm{h}}{\sim}$ ($\mathcal{B}$ unital) and the inclusion

$$\mathcal{U}(\mathsf{M}_n(\mathcal{B})) \hookrightarrow \mathcal{U}(\mathsf{M}_{n+1}(\mathcal{B}))$$

induces an inclusion

$$(\mathcal{U}(\mathsf{M}_n(\mathcal{B}))/\underset{\mathrm{h}}{\sim}) \to (\mathcal{U}(\mathsf{M}_{n+1}(\mathcal{B}))/\underset{\mathrm{h}}{\sim})$$

and

$$(\mathcal{U}_\infty(\mathcal{B})/\underset{\mathrm{h}}{\sim}) = \lim_{\underset{n}{\to}}(\mathcal{U}(\mathsf{M}_n(\mathcal{B}))/\underset{\mathrm{h}}{\sim})$$

Q: When does

$$\lim_{\underset{n}{\to}}(\mathcal{U}(\mathsf{M}_n(\mathcal{B}))/\underset{\mathrm{h}}{\sim}) = (\mathcal{U}_{n_0}(\mathcal{B})/\underset{\mathrm{h}}{\sim})$$

for some fixed n_0? Generally a difficult question. For now we believe that $n_0 = 1$ for $C(\mathrm{S}^1)$. If $f : \mathrm{S}^1 \to \mathbb{C}$ is continuous and unitary, $\exists\, g : [0,1] \to \mathbb{R}$ such that $g(1) - g(0) \in 2\pi\mathbb{Z}$, $f = \mathrm{e}^{ig}$ then the winding number is $W(f) = \frac{1}{2\pi}(g(1) - g(0))$:

$$\mathcal{U}(C(\mathrm{S}^1)) \cong \mathbb{Z}$$

On $C^1(\mathrm{S}^1) \overset{\text{dense}}{\subset} C(\mathrm{S}^1)$, W is given by

$$W([f]) = \frac{1}{2\pi i}\int \frac{1}{f}\,\mathrm{d}\,f$$

If $\mathcal{B}$ is commutative, so that $\mathcal{B} = C_0(X)$, X locally compact Hausdorff, let

$$\mathcal{U}(\mathsf{M}_n(\mathcal{B}^+)) \overset{\substack{\text{pt-wise}\\ \det}}{\longrightarrow} \mathcal{B}^+ = C(X^+)$$

By construction of the embedding $\mathcal{U}(\mathsf{M}_n(\mathcal{B})) \overset{i_n}{\hookrightarrow} \mathcal{U}(\mathsf{M}_{n+1}(\mathcal{B}))$, $\det(i_n(b)) = \det(b)$; and $\det : \mathcal{U}_\infty(\mathcal{B}^+) \to \mathcal{U}(\mathcal{B}^+)$ is continuous, hence inducing

$$\Delta : K_1(\mathcal{B}) \to \mathcal{U}(\mathcal{B}^+)/\underset{\mathrm{h}}{\sim} \quad (n_0 = 1)\ ;\ \text{a group homomorphism}$$

$$0 \to \operatorname{Ker}\Delta \to K_1(\mathcal{B}) \underset{s}{\rightleftarrows} (\mathcal{U}(\mathcal{B}^+)/\underset{\mathrm{h}}{\sim}) \to 0 \quad \text{split exact}$$

$$\text{because } K_1(\mathcal{B}) = \varinjlim_n \mathcal{U}(\mathsf{M}_n(\mathcal{B}^+))/\underset{\mathrm{h}}{\sim}$$

So if $\mathcal{B}$ is commutative, $K_1(\mathcal{B}) \cong \operatorname{Ker}\Delta \oplus (\mathcal{U}(\mathcal{B}^+)/\underset{\mathrm{h}}{\sim})$.

$\operatorname{Ker}\Delta$ is not always 0; $X^+ = \mathbb{T}^3$ is a counterexample.

Connecting the half-exact sequences:

$$\begin{tikzcd}
K_0(\mathcal{J}) \arrow[r, "K_0(i)"] & K_0(\mathcal{E}) \arrow[r, "K_0(q)"] & K_0(\mathcal{A}) \arrow[dll, "\exp"] \\
K_1(\mathcal{J}) \arrow[r, "K_1(i)"] & K_1(\mathcal{E}) \arrow[r, "K_1(q)"] & K_1(\mathcal{A}) \arrow[ull, "\mathrm{index}"]
\end{tikzcd}$$

Figure 6: Boundary maps

The index map: connects K_1(quotient) to K_0(ideal). If the SES splits

$$0 \to \mathcal{J} \overset{i}{\to} \mathcal{E} \underset{s}{\overset{q}{\rightleftarrows}} \mathcal{A} \to 0 \ \text{(supposing } \mathcal{A} \text{ and } \mathcal{E} \text{ are unital and } q \text{ is unital)}$$

then $K_1(q)$ is surjective (and $K_0(i)$ is injective), so in this case we expect the index map to be 0: $\operatorname{Im} K_1(q) = K_1(\mathcal{A}) = \operatorname{Ker}(\text{index map})$. Furthermore, in this case we can lift any unitary from $\mathcal{A}$ to $\mathcal{E}$: if $u \in \mathcal{A}$ is unitary, then in the lift $[s(u) + (1 - s(1))]$ of u, $s(u)$ is not necessarily unitary, but $s(1)$ is a projection and $1 - s(1)$ is a projection orthogonal to $\operatorname{Im} s$, and therefore

$$[s(u)^* + (1 - s(1))][s(u) + (1 - s(1)] = s(1) + (1 - s(1)) = 1$$

so the lift is unitary.

So we expect $\operatorname{ind}[u]_1 \neq 0$ only if there is no unitary lift of u.

First attempt to define index.

Suppose we can lift u to a partial isometry $w \in \mathcal{E}$; w is a partial isometry if $w^*ww^* = w^*$ and $ww^*w = w$. These relations imply that $(ww^*)^2 = ww^*$ and $(w^*w)^2 = w^*w$, which means they are both projections. If $\mathcal{E} \subset \mathcal{B}(\mathcal{H})$, then w^*w is the domain projection of a unitary $\tilde{w} : w^*w\mathcal{H} \to ww^*\mathcal{H}$ and ww^* is the image projection.

So suppose we can lift u to a partial isometry w: $q(w) = u$. Then $q(1 - w^*w) = 1 - u^*u = 0$ and likewise $q(1 - ww^*) = 0$, that is,

$$1 - w^*w\,,\ 1 - ww^* \ \in i(\mathcal{J}) = \operatorname{Ker} q$$

Then we define[15]

$$\operatorname{ind}[u]_1 = [1 - w^*w]_0 - [1 - ww^*]_0$$

This should be shown independent of choice of w (for a given u) and independent of the choice of u in $[u]_1$.

[15] If $i(\mathcal{J})$ is identified with the ideal $\mathcal{J}$, then $1-w^*w$ and $1-ww^*$ belong to $\mathcal{J}$ and determine elements of $K_0(\mathcal{J})$.

Ex.:

$$0 \to \mathcal{K}(\mathcal{H}) \xrightarrow{i} \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to 0$$

K-theory applied to this looks like this:

$$\begin{tikzcd}
K_1(\mathcal{K}(\mathcal{H})) \arrow[r, "K_1(i)"] & K_1(\mathcal{B}(\mathcal{H})) \arrow[r, "K_1(q)"] & K_1(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})) \arrow[dll, rounded corners, "\text{index}", "\text{index}"'] \\
K_0(\mathcal{K}(\mathcal{H})) \arrow[r, "K_0(i)"] & K_0(\mathcal{B}(\mathcal{H})) \arrow[r, "K_0(q)"] & K_0(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}))
\end{tikzcd}$$

Using the isomorphism $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$ induced by the trace, and earlier examples:

$$\begin{tikzcd}
0 \arrow[r, "K_1(i)"] & 0 \arrow[r, "K_1(q)"] & K_1(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})) \arrow[dll, rounded corners, "\text{index}", "\text{index}"'] \\
\mathbb{Z} \arrow[r, "K_0(i)"] & 0 \arrow[r, "K_0(q)"] & K_0(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}))
\end{tikzcd}$$

Figure 7: Calkin algebra example

Theorem (Atkinson)**.** *$F \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator, i.e.,* $\operatorname{Ker} F$ & $\operatorname{Coker} F$ *are finite dimensional $\Longleftrightarrow q(F)$ is invertible.*

If $u \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is unitary then u has a lift w which is a Fredholm operator, $\operatorname{ind}[u]_1 = [1 - w^*w]_0 - [1 - ww^*]_0$, and

$$\operatorname{Tr}([1 - w^*w]_0 - [1 - ww^*]_0) = \dim \operatorname{Ker} w - \dim \operatorname{Coker} w$$

29 (8 April 2009)

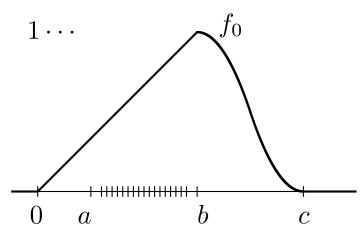
$K_1(\mathcal{A}) \overset{\text{ind}}{\to} K_0(\mathcal{J})$ (continued)

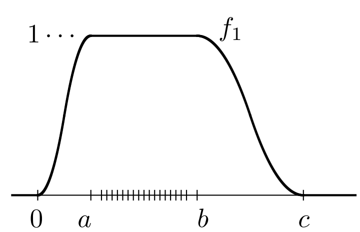
Lemma. *If F is Fredholm, then there exists a partial isometry W such that $F \underset{\mathrm{h}}{\sim} W$ (as Fredholm)*

Proof. $F = W|F|$ (polar decomposition) where $|F| = \sqrt{F^*F}$ and W is a partial isometry. Since F is Fredholm, 0 is (if at all) an isolated value in $\sigma(|F|)$ of finite multiplicity. So $(\sigma(|F|) \setminus \{0\}) \subset [a, b]$, $a > 0$, and the following homotopy brings F to W

$$|F| \underset{\mathrm{h}}{\sim} 1 - P(F)_{\text{eigenvalue } 0}$$

In greater detail: if $f_0, f_1 \in C([0, c], \mathbb{R})$ as shown below, then $f_0 \underset{\mathrm{h}}{\sim} f_1$ and $f_0(|F|) = |F|$.





∎

Back to the index map. If

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$$

is such that $\mathcal{E}$ is unital ($\Rightarrow \mathcal{A}$ is unital and q is unital), and if $u \in \mathcal{U}(\mathsf{M}_n(\mathcal{A}))$ admits a lift W which is a partial isometry in $\mathsf{M}_n(\mathcal{E})$, then

$$\operatorname{ind}[u]_1 = [i^{-1}(1 - W^*W)]_0 - [i^{-1}(1 - WW^*)]_0$$

Remark. One cannot always find a partial isometry lift of a unitary.

Other “standard” picture for ind (cf. Bruce Blackadar, *K-Theory for Operator Algebras*)

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$$

Not supposing that $\mathcal{E}$ is unital: will add units

$$\underset{=\mathcal{J}\times\mathbb{C}}{\mathcal{J}^+} \xrightarrow{i^+} \mathcal{E}^+ \xrightarrow{q^+} \mathcal{A}^+\,, \quad i^+(j,\lambda) = (i(j),\lambda)$$

$$i^+ \text{ is injective; } q^+ \text{ is surjective; } \operatorname{Ker} q^+ \subset \operatorname{Im} i^+\,;$$
$$\operatorname{Im} i^+ = \{x \in \mathcal{E}^+ \mid q^+(x) \in \{0\} \times \mathbb{C} \subset \mathcal{A}^+\}$$

No longer a SES, but there is a commutative diagram with projections $\pi_{\mathcal{J}}(j,\lambda) = \lambda$, etc.,

$$\begin{array}{ccccc} \mathcal{J}^+ & \xrightarrow{i^+} & \mathcal{E}^+ & \xrightarrow{q^+} & \mathcal{A}^+ \\ \downarrow \pi_{\mathcal{J}} & & \downarrow \pi_{\mathcal{E}} & & \downarrow \pi_{\mathcal{A}} \\ \mathbb{C} & \xrightarrow{\mathrm{id}} & \mathbb{C} & \xrightarrow{\mathrm{id}} & \mathbb{C} \end{array}$$

Lemma. *Any $u \in \mathcal{U}(\mathsf{M}_n(\mathcal{A}^+))_0$ admits a unitary lift*

$$(\Leftarrow q^+(\,\mathcal{U}(\mathsf{M}_n(\mathcal{E}^+))_0\,) = \mathcal{U}(\mathsf{M}_n(\mathcal{A}^+))_0)$$

Proof. Hints:

i) (Takes a while.) If $u \in \mathcal{U}(\mathsf{M}_n(\mathcal{A}^+))_0$, then $\exists\, k_1, \ldots, k_n \in \mathsf{M}_n(\mathcal{A}^+)^{\text{s.a.}}$ ($k_i = k_i^*$) such that $u = \mathrm{e}^{ik_1} \cdots \mathrm{e}^{ik_n}$. (Factors of which one can take logs.)

ii) A s.a. element always admits a s.a. lift: if $x \in \mathsf{M}_n(\mathcal{A}^+)$ and $k \in \mathsf{M}_n(\mathcal{E}^+)$ is such that $q^+(k) = x$, then $q^+(k^*) = x^* = x$, and so $\frac{1}{2}(k^* + k)$ is a s.a. lift of x.

□

Other definition of ind: Let $u \in \mathcal{U}(\mathsf{M}_n(\mathcal{A}^+))$. Then $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in \mathcal{U}(\mathsf{M}_{2n}(\mathcal{A}^+))_0$ (by the Whitehead lemma).

Let $W \in \mathcal{U}(\mathsf{M}_{2n}(\mathcal{E}^+))_0$ be such that $q^+(W) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. $W \begin{pmatrix} 1_n & 0_n \\ 0_n & 0_n \end{pmatrix} W^*$ is a projection in $\mathsf{M}_{2n}(\mathcal{E}^+)$, and

$$q^+\left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathsf{M}_{2n}(\{0\} \times \mathbb{C} \subset \mathcal{A}^+) \Rightarrow \text{ (by commuting diagram)}$$

$$\pi_{\mathcal{J}} \left(\underbrace{(i^+)^{-1}\left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right)}_{\in \mathsf{M}_{2n}(\mathcal{J}^+)} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$= \pi_{\mathcal{A}} \circ q^+ \left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^* - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0 \quad [!]$$

Hence

$$\left[(i^+)^{-1}\left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right)\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

is an element of $K_0(\mathcal{J})$.

Definition.

$$\mathrm{ind}[u]_1 = \left[(i^+)^{-1}\left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right)\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

$$\begin{tikzcd} K_1(\mathcal{J}) \arrow[r, "K_1(i)"] & K_1(\mathcal{E}) \arrow[r, "K_1(q)"] & K_1(\mathcal{A}) \arrow[dll, "\mathrm{ind}"] \\ K_0(\mathcal{J}) \arrow[r, "K_0(i)"] & K_0(\mathcal{E}) \arrow[r, "K_0(q)"] & K_0(\mathcal{A}) \end{tikzcd}$$

Theorem.

i) Independent of choice of W;

ii) Independent of choice of representative for $[u]_1$;

iii) ind *is a group homomorphism;*

iv) $\mathrm{Im}\, K_1(q) \subset \mathrm{Ker}\, \mathrm{ind}$*;*

v) $\mathrm{Im}\, \mathrm{ind} \subset \mathrm{Ker}\, K_0(i)$

vi) Equalities in iv) and v).

Proof. i) Let W' be another unitary lift of $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$.

$$q^+(WW^*) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = 1 \in \mathsf{M}_{2n}(\{0\} \times \mathbb{C} \subset \mathcal{A}^+)$$

So $W'W^* \in \operatorname{Im} i^+$ and $\exists\, z \in \mathsf{M}_{2n}(\mathcal{J}^+)$ such that $i^+(W'W^*) = z$; z is unitary since the pre-image of a unitary is unitary.

$$z\ (i^+)^{-1} \left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^* \right)\ z^* = (i^+)^{-1} \left(W' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W'^* \right)$$

so the two expressions

$$(i^+)^{-1} \left(W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^* \right) \text{ and } (i^+)^{-1} \left(W' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W'^* \right)$$

are $\underset{\mathrm{h}}{\sim}$.

ii) Let $u_1 \underset{\mathrm{h}}{\sim} u_2$ in $\mathcal{U}(\mathsf{M}_n(\mathcal{A}^+))$; hence $u_1 u_2^* \underset{\mathrm{h}}{\sim} 1$ and $u_1^* u_2 \underset{\mathrm{h}}{\sim} 1$; hence $\exists\, a, b \in \mathcal{U}(\mathsf{M}_\mathsf{n}(\mathcal{E}^+))$ such that $q^+(a) = u_1^* u_2$ and $q^+(b) = u_1 u_2^*$. Let $W_1 \in \mathcal{U}(\mathsf{M}_{2n}(\mathcal{E}^+))$ be a lift of $\begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix}$ and set $W_2 = W_1 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{U}(\mathsf{M}_{2n}(\mathcal{E}^+))$.

$$q^+(W_2) = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix} \begin{pmatrix} u_1^* u_2 & 0 \\ 0 & u_1 u_2^* \end{pmatrix} = \begin{pmatrix} u_2 & 0 \\ 0 & u_2^* \end{pmatrix}$$

$$\begin{aligned} W_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_2^* &= W_1 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & 0 \\ 0 & b^* \end{pmatrix} W_1^* \\ &= W_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_1^* \end{aligned}$$

The two choices yield the same element in $K_0(\mathcal{J})$.

iii) Rather simple (the group operation is the direct, or diagonal, sum).

iv) If $[u]_1 \in \operatorname{Im} K_1(q)$, then u has a unitary lift V. Then $\exists\, v$ such that $v \underset{\mathrm{h}}{\sim} u$

$$\begin{aligned} \operatorname{ind}[u]_1 &= \operatorname{ind}[v]_1 \\ &= \left[(i^*)^{-1} \begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^* & 0 \\ 0 & V \end{pmatrix} \right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]_0 \\ &= 0 \end{aligned}$$

and

$$K_0(i^+) \left(\left[(i^+)^{-1} W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^* \right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]_0 \right) \overset{[!]}{=} 0$$

since they are unitarily equivalent in $\mathsf{M}_{2n}(\mathcal{E}^+)$! ☐

30 (13 April 2009)

Bott map and higher K-groups

Comments on the proofs of iv), v) and vi) of the theorem from the previous lecture

$$(*) \qquad \mathrm{ind}[u]_1 = \big[(i^+)^{-1}(1-v^*v)\big]_0 - \big[(i^+)^{-1}(1-vv^*)\big]_0 \in K_0(\mathcal{J})$$

provided that u has a partial isometry lift v.

Note: not every unitary $u \in \mathsf{M}_n(\mathcal{A}^+)$ admits a partial isometry lift in $\mathsf{M}_n(\mathcal{E}^+)$, but $\exists\, k$ such that $\begin{pmatrix} u & 0 \\ 0 & 0_k \end{pmatrix}$ admits a partial isometry lift in $\mathsf{M}_{n+k}(\mathcal{E}^+)$ and then the same formula holds.

Let us see that $(*)$ follows from the formula

$$\mathrm{ind}[u]_1 = \left[(i^+)^{-1} W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

where $W \in \mathsf{M}_{2n}(\mathcal{E}^+)$ is a unitary lift of $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. Suppose v is a partial isometry lift of u, $v \in \mathsf{M}_n(\mathcal{E}^+)$ (1 is the added unit; 1 also denotes 1_n)

$$W := \begin{pmatrix} v & 1-vv^* \\ 1-v^*v & v^* \end{pmatrix}$$

$$\text{(easy:[16])} \ \ WW^* = W^*W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$q^+(W) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

So $\mathrm{ind}[u]_1$ can be written with this W.

$$W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^* = \begin{pmatrix} 1-(1-vv^*) & 0 \\ 0 & 1-v^*v \end{pmatrix}, \quad \Rightarrow$$

$$\left[(i^+)^{-1} W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right]_0 = \left[(i^+)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1-v^*v \end{pmatrix}\right]_0 + \left[{i^+}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$
$$- \left[{i^+}^{-1}\begin{pmatrix} 1-vv^* & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$
$$= \underbrace{\left[{i^+}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0}_{\substack{=\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0 \\ \text{subtract from both sides}}} + \left[{i^+}^{-1}(1-v^*v)\right]_0 - \left[{i^*}^{-1}(1-vv^*)\right]_0$$

[16] v is a partial isometry iff $v^*vv^* = v^*$ and $vv^*v = v$. See the 6 April lecture.

and $(*)$ follows.

For the proof of vi), see the book of Rørdam[17].

Summary: we have an exact sequence:

$$K_1(\mathcal{J}) \overset{K_1(i)}{\to} K_1(\mathcal{E}) \overset{K_1(q)}{\to} K_1(\mathcal{A}) \xrightarrow{\text{ind}} K_0(\mathcal{J}) \overset{K_0(i)}{\to} K_0(\mathcal{E}) \overset{K_0(q)}{\to} K_0(\mathcal{A})$$

Ex.: If $K_i(\mathcal{E})$, $i = 0, 1$, is trivial then ind is an isomorphism!

1) $\mathcal{E} = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ an infinite dimensional, separable Hilbert space, then $K_i(\mathcal{E}) = 0$, $i = 0, 1$, so

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to 0$$
$$\Rightarrow K_1(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})) \overset{\text{ind}}{\cong} K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$$

2) $\mathcal{E} = \mathrm{C}\,\mathcal{A}$, the cone of the algebra $\mathcal{A}$. $\mathrm{C}\,\mathcal{A}$ is contractible, so $K_i(\mathcal{E}) = 0$, $i = 0, 1$, and

$$0 \to \underset{\text{suspension}}{\mathrm{S}\,\mathcal{A}} \to \underset{\text{cone}}{\mathrm{C}\,\mathcal{A}} \xrightarrow{\text{ev}_1} \mathcal{A} \to 0$$

yields an important isomorphism:

$$\Theta_{\mathcal{A}} = \text{ind} : K_1(\mathcal{A}) \to K_0(\mathrm{S}\,\mathcal{A})$$

<u>The suspension functor</u>

$$\mathrm{S} : \text{Banach} *\text{-algs} \to \text{Banach} *\text{-algs}$$
$$\mathcal{A} \mapsto \mathrm{S}\,\mathcal{A} = \{f : [0,1] \to \mathcal{A} \mid f \text{ is conts; } f(0) = f(1) = 0\}$$

If $\mathcal{A} \overset{g}{\to} \mathcal{B}$ is a morphism, then $\mathrm{S}\,\mathcal{A} \overset{\mathrm{S}(g)}{\to} \mathrm{S}\,\mathcal{B}$, with $(\mathrm{S}(g)a)(t) = g(a(t))$.

S is exact:

$$0 \to \mathcal{J} \overset{i}{\to} \mathcal{E} \overset{q}{\to} \mathcal{A} \to 0 \text{ is SES}$$
$$\Rightarrow 0 \to \mathrm{S}\,\mathcal{J} \overset{\mathrm{S}(i)}{\to} \mathrm{S}\,\mathcal{E} \overset{\mathrm{S}(q)}{\to} \mathrm{S}\,\mathcal{A} \to 0 \text{ is SES}$$

So $K_0 \circ \mathrm{S}$ is a functor which is "equal" to K_1.

Θ is natural.

$$\mathcal{A} \overset{f}{\to} \mathcal{B} \quad \Rightarrow$$

$$\begin{array}{ccc} K_1(\mathcal{A}) & \xrightarrow{K_1(f)} & K_1(\mathcal{B}) \\ \Theta_{\mathcal{A}} \downarrow & & \downarrow \Theta_{\mathcal{B}} \\ K_0(\mathcal{A}) & \xrightarrow{K_0(f)} & K_0(\mathcal{B}) \end{array}$$

[17] *An Introduction to K-theory for C*-algebras*, by Mikael Rørdam, Flemming Larsen, and Niels Jakob Laustsen.

is a commuting diagram.

The index map ind is natural. If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{q} & \mathcal{A} & \longrightarrow & 0 \\ & & \phi\downarrow & & \downarrow\psi & & \downarrow\alpha & & \\ 0 & \longrightarrow & \mathcal{J}' & \xrightarrow{i'} & \mathcal{E}' & \xrightarrow{q'} & \mathcal{A}' & \longrightarrow & 0 \end{array}$$

are two SESs in a commuting diagram, then

$$\begin{array}{ccc} K_1(\mathcal{A}) & \xrightarrow{\mathrm{ind}} & K_0(\mathcal{J}) \\ K_1(\alpha)\downarrow & & \downarrow K_0(\phi) \\ K_1(\mathcal{A}') & \xrightarrow{\mathrm{ind}'} & K_0(\mathcal{J}') \end{array}$$

commutes.

Higher degree K-groups.

Definition. $K_n(\mathcal{A}) = K_0(\mathrm{S}^n\mathcal{A})$, $K_n(f) = K_0(\mathrm{S}^n(f))$.

This gives a long exact sequence. If $0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$ is a SES,

$$\cdots \xrightarrow{\delta_{n+1}} K_n(\mathcal{J}) \xrightarrow{K_n(i)} K_n(\mathcal{E}) \xrightarrow{K_n(q)} K_n(\mathcal{A}) \xrightarrow{\delta_n} K_{n-1}(\mathcal{J}) \xrightarrow{K_{n-1}(i)}$$
$$\xrightarrow{K_{n-1}(i)} K_{n-1}(\mathcal{E}) \xrightarrow{K_{n-1}(q)} \cdots \xrightarrow{K_0(q)} K_0(\mathcal{A})$$

is an exact sequence (no claim is made for $\mathrm{Im}\, K_0(q)$), where δ_n is defined by the commuting diagram

$$\begin{array}{ccc} K_n(\mathcal{A}) & \xrightarrow{\delta_n} & K_{n-1}(\mathcal{J}) \\ \cong\downarrow & & \\ K_0(\mathrm{S}(\mathrm{S}^{n-1}\mathcal{A})) & & \downarrow = \\ \Theta^{-1}_{\mathrm{S}^{n-1}\mathcal{A}}\downarrow\cong & & \\ K_1(\mathrm{S}^{n-1}\mathcal{A}) & \xrightarrow{\mathrm{ind}} & K_0(\mathrm{S}^{n-1}\mathcal{J}) \end{array}$$

that is, $\delta_n = \mathrm{ind} \circ \Theta^{-1}_{\mathrm{S}^{n-1}\mathcal{A}}$ ($\delta_1 = \mathrm{ind}$).

Bott periodicity

The Bott map $\beta_{\mathcal{A}} : K_0(\mathcal{A}) \to K_1(\mathrm{S}\mathcal{A})$

Definition.

$$\beta_{\mathcal{A}}([p]_0) := [t \mapsto \mathrm{e}^{2\pi i t p}]_1 \,; \;\; p \in \mathrm{Proj}\, \mathsf{M}_n(\mathcal{A})\,,\; \mathrm{e}^{2\pi i t p} \in \mathsf{M}_n(\mathcal{A})$$

where

$$\begin{aligned} e^{2\pi i t p} &= \sum_{\nu=0}^{\infty} \frac{(2\pi i t)^{\nu}}{\nu!}\, p^{\nu} \qquad\qquad p^2 = p \;\Rightarrow\; p^{\nu} = \begin{cases} p & \nu \geq 1 \\ 1 & \nu = 0 \end{cases} \\ &= \sum_{\nu=0}^{\infty} \frac{(2\pi i t)^{\nu}}{\nu!}\, p + (1-p) \\ &= e^{2\pi i t}\, p + (1-p) \in \mathcal{U}(\mathsf{M}_n(\mathcal{A})) \end{aligned}$$

and

$$e^{2\pi i 0} p + (1-p) = e^{2\pi i 1} p + (1-p) = 1$$

so

$$(t \mapsto e^{2\pi i t p}) \in \mathcal{U}(\mathsf{M}_n((\mathrm{S}\mathcal{A})^+))$$

Lemma.

1) *$\beta_{\mathcal{A}}$ is well-defined;*
2) *$\beta_{\mathcal{A}}$ is a group homomorphism;*
3) *$\beta_{\mathcal{A}}$ is natural:*

$$\begin{array}{ccc} K_0(\mathcal{A}) & \xrightarrow{K_0(f)} & K_0(\mathcal{B}) \\ \beta_{\mathcal{A}} \downarrow & & \downarrow \beta_{\mathcal{B}} \\ K_1(\mathrm{S}\mathcal{A}) & \xrightarrow{K_1(\mathrm{S}f)} & K_1(\mathrm{S}\mathcal{B}) \end{array}$$

commutes.

Proof. Regarding 1), if $p \underset{\mathrm{h}}{\sim} q$, then the homotopy $(p_s)_{s\in[0,1]}$, with $p_0 = p$ and $p_1 = q$, yields a homotopy

$$s \mapsto (t \mapsto e^{2\pi i t p_s})$$

between the two results. $\square$

31 (15 April 2009)

Bott theorem

Definition (Recap from the previous lecture)**.** (For unital $\mathcal{A}$)

$$\begin{aligned}\beta_{\mathcal{A}} &: K_0(\mathcal{A}) \to K_1(\mathrm{S}\mathcal{A})\\ \beta_{\mathcal{A}}([p]_0) &= [f_p]_1\,,\ \text{where } f_p : [0,1] \to \mathcal{A}\,,\ \text{continuous}\\ f_p(t) &= \mathrm{e}^{2\pi i t p} = \mathrm{e}^{2\pi i t}\, p + (1-p)\end{aligned}$$

In particular, $f_p(0) = f_p(1) = 1$, so $f_p \in (\mathrm{S}\mathcal{A})^+$.

$\beta_{\mathcal{A}}$ is a well-defined homomorphism of groups which is natural. In particular,

$$\begin{aligned}\beta_{\mathcal{A}}([0]_0) &= [1]_1\\ \beta_{\mathcal{A}}([1]_0) &= [(t \mapsto \mathrm{e}^{2\pi i t}\, 1)]_1\end{aligned}$$

If $\mathcal{A} = \mathbb{C}$, so that $[1]_0$ is the generator of $K_0(\mathbb{C})$, then[18]

$$\beta_{\mathcal{A}}([1]_0) = [(t \mapsto \mathrm{e}^{2\pi i t})]_1$$

is a non-trivial element in $K_1(C(\mathrm{S}^1))$. The non-triviality was measured by the winding number of f_1 which is 1.

So $\beta_{\mathbb{C}} : K_0(\mathbb{C}) \to K_1(C(\mathrm{S}^1))$ (the most natural map).

$\beta_{\mathcal{A}}$ on non-unital algebras. For $\mathcal{A}$ non-unital,

$$0 \to \mathcal{A} \to \mathcal{A}^+ \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0 \quad \text{(split SES)}$$

$$\begin{array}{rccccccl} \rightsquigarrow & 0 \to & K_0(\mathcal{A}) \to & K_0(\mathcal{A}^+) & \overset{K_0(\pi)}{\to} & K_0(\mathbb{C}) & \to 0 & \text{(split SES)}\\ \text{(apply suspension)} & & \beta_{\mathcal{A}} \downarrow & \beta_{\mathcal{A}^+} \downarrow & & \downarrow \beta_{\mathbb{C}} & & \\ & 0 \to & K_1(\mathrm{S}\mathcal{A}) \to & K_1(\mathrm{S}\mathcal{A}^+) & \overset{K_1(\mathrm{S}\pi)}{\to} & K_1(\mathrm{S}\mathbb{C}) & \to 0 & \end{array}$$

and $\beta_{\mathcal{A}}$ is defined by the commuting diagram.

For non-unital $\mathcal{A}$

$$\begin{aligned}\beta_{\mathcal{A}} &: K_0(\mathcal{A}) \to K_1(\mathrm{S}\mathcal{A})\\ \beta_{\mathcal{A}}([p]_0 - [1_n]_0) &= f_p f_{1_n}^* = [(t \mapsto \mathrm{e}^{2\pi i t p} \mathrm{e}^{-2\pi i t 1_n})]_1\\ &\text{where } p, 1_n \in \mathrm{Proj}(\mathsf{M}_n(\mathcal{A}^+))\end{aligned}$$

[18] We have $(\mathrm{S}\,\mathbb{C})^+ \cong C(\mathrm{S}^1)$, and identifying $[0,1]/\{0,1\}$ with S^1, $(t \mapsto \mathrm{e}^{2\pi i t}) \in C(\mathrm{S}^1)$ corresponds to the identity map $id : \mathrm{S}^1 \to \mathrm{S}^1$.

Theorem (Bott)**.** *For all Banach $*$-algebras $\mathcal{A}$, $\beta_{\mathcal{A}}$ is an isomorphism.*

Crucial ingredient in the proof: the algebras are closed under the holomorphic functional calculus. $\square$

Corollary. *Bott theorem* $\Leftrightarrow \forall\, n\,,\ \underbrace{K_{n+2}(\mathcal{A}) = K_n(\mathcal{A})}_{\textit{Bott periodicity}}$ *(for all Banach $*$-algebras).*

Proof.

$$K_n(\mathcal{A}) := K_0(\mathrm{S}^n\mathcal{A}) \underset{\substack{\text{Bott}\\ \beta_{\mathrm{S}^n\mathcal{A}}}}{\cong} K_1(\mathrm{S}^{n+1}\mathcal{A}) \underset{\Theta_{\mathrm{S}^{n+1}\mathcal{A}}}{=} K_0(\mathrm{S}^{n+2}\mathcal{A}) =: K_{n+2}(\mathcal{A})$$

$\square$

Corollary. *Any SES* $0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \to 0$ *yields a* 6*-term ES:*

$$\begin{tikzcd}
K_0(\mathcal{J}) \arrow[r, "K_0(i)"] & K_0(\mathcal{E}) \arrow[r, "K_0(q)"] & K_0(\mathcal{A}) \arrow[dll, bend left, "\exp"] \\
K_1(\mathcal{J}) \arrow[r, "K_1(i)"] & K_1(\mathcal{E}) \arrow[r, "K_1(q)"] & K_1(\mathcal{A}) \arrow[ull, bend right, "\operatorname{ind}"']
\end{tikzcd}$$

Figure 8: 6-term ES. $\exp := \Theta_{\mathcal{J}}^{-1} \circ \operatorname{ind}_{\mathrm{S}} \circ \beta_{\mathcal{A}}$

Proof. The map exp is defined by the commuting diagram[19]

$$\begin{tikzcd}
K_0(\mathcal{E}) \arrow[r, "K_0(q)"] \arrow[d, "\beta_{\mathcal{E}}"'] & K_0(\mathcal{A}) \arrow[r, dotted, "\exp"] \arrow[d, "\beta_{\mathcal{A}}"] & K_1(\mathcal{J}) \arrow[r, "K_1(i)"] \arrow[d, "\Theta_{\mathcal{J}}"'] & K_1(\mathcal{E}) \arrow[d, "\Theta_{\mathcal{E}}"] \\
K_1(\mathrm{S}\mathcal{E}) \arrow[r, "K_1(q)"] & K_1(\mathrm{S}\mathcal{A}) \arrow[r, "\operatorname{ind}_{\mathrm{S}}"] & K_0(\mathrm{S}\mathcal{J}) \arrow[r, "K_0(i)"] & K_0(\mathrm{S}\mathcal{E})
\end{tikzcd}$$

where $\operatorname{ind}_{\mathrm{S}}$ is the index map from the SES

$$0 \to \mathrm{S}\mathcal{J} \to \mathrm{S}\mathcal{E} \to \mathrm{S}\mathcal{A} \to 0$$

$\square$

[19] The left and right squares commute by the naturality of β and Θ.

Proposition. *For* $p \in \mathrm{Proj}\,\mathsf{M}_n(\mathcal{A}^+)$, $\exp([p]_0 - [1_n]_0) = [{i^+}^{-1}\,\mathrm{e}^{-2\pi i h}]_1$, *where* h *is a self-adjoint lift of* p *in* $\mathsf{M}_n(\mathcal{E}^+)$.

($\mathrm{e}^{-2\pi i h}$ is in $\mathrm{Im}\, i^+$ because $q^+(\mathrm{e}^{-2\pi i h}) = \mathrm{e}^{-2\pi i p} = 1_n$; because $\mathrm{spec}(p) = \{0, 1\}$.)

Exercise: show that this is independent of the choice of p and the choice of h (linear homotopy between choices of h; unitary equivalence of projections).

Proof. We will do this in the unital case: suppose that $\mathcal{E}$, $\mathcal{A}$ and q are unital. We need to show that

$$\Theta_{\mathcal{J}}[{i^+}^{-1}\,\mathrm{e}^{-2\pi i h}]_1 = \mathrm{ind}_{\mathrm{S}}[f_p]_1$$

with $f_p \in \mathcal{U}_n((\mathrm{S}\mathcal{A})^+)$, $f_p(t) = \mathrm{e}^{2\pi i t p} \in \mathcal{U}_n(\mathcal{A})$.

To calculate $\mathrm{ind}_{\mathrm{S}}[f_p]_1$ we need a unitary lift of $\begin{pmatrix} f_p & 0 \\ 0 & f_p^* \end{pmatrix}$; call it $V \in \mathcal{U}\mathsf{M}_{2n}(\mathrm{S}\mathcal{E})$. So $V : [0,1] \to \mathcal{U}_{2n}(\mathcal{E})$. Then

$$\mathrm{ind}_{\mathrm{S}}([f_p]_1) = \left[{i^+}^{-1}\, V \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} V^*\right]_0 - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

Let h be a self-adjoint lift of p in $\mathsf{M}_n(\mathcal{E})$, and let

$$z : [0,1] \to \mathcal{E}\,,\ z(t) = \mathrm{e}^{2\pi i t h} \in \mathcal{U}(\mathsf{M}_n(\mathcal{E}))$$

Then

$$q\left(V(t)\begin{pmatrix} z^*(t) & 0 \\ 0 & z(t) \end{pmatrix}\right) = \begin{pmatrix} f_p(t) & 0 \\ 0 & f_p^*(t) \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-2\pi i t p} & 0 \\ 0 & \mathrm{e}^{2\pi i t p} \end{pmatrix} = 1_{2n}$$

hence $V(t)\begin{pmatrix} z^*(t) & 0 \\ 0 & z(t) \end{pmatrix} \in \mathrm{Im}\, i^+$. Define

$$W(t) = {i^+}^{-1}\left(V(t)\begin{pmatrix} z^*(t) & 0 \\ 0 & z(t) \end{pmatrix}\right)$$

$$\begin{aligned} \implies \quad & i^+(W(0)) = \underbrace{V(0)}_{=1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2n} \\ \text{and} \quad & i^+(W(1)) = \underbrace{V(1)}_{=1}\begin{pmatrix} \mathrm{e}^{-2\pi i h} & 0 \\ 0 & \mathrm{e}^{2\pi i h} \end{pmatrix} = 1_{2n} \end{aligned}$$

This tells us that $(W(t))_t$ is a homotopy in $\mathcal{U}_{2n}(\mathcal{J}^+)$ between 1 and $\begin{pmatrix} \mathrm{e}^{-2\pi i h} & 0 \\ 0 & \mathrm{e}^{2\pi i h} \end{pmatrix}$. $\Theta_{\mathcal{J}}$ was the index function for $0 \to \mathrm{S}\mathcal{J} \to \mathrm{C}\mathcal{J} \overset{\mathrm{ev}_1}{\to} \mathcal{J} \to 0$, therefore

$$\Theta_{\mathcal{J}}[{i^+}^{-1}\,\mathrm{e}^{-2\pi i h}]_1 = \left[W\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

but

$$i^+\left(W\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}W^*\right) = V\begin{pmatrix}z^* & 0\\ 0 & z\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & z^*\end{pmatrix}V^* = V\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}v^*$$

$\square$

32 (17 April 2009)

Proof[20] of the Bott theorem, part 1.

Recall: Any C*-algebra $\mathcal{A}$

$$K_0(\mathcal{A}) \cong K_0(\mathrm{S}^2\mathcal{A}) \text{ with isomorphism given by } \Theta_{\mathrm{S}\mathcal{A}} \circ \beta_{\mathcal{A}}$$
$$K_1(\mathcal{A}) \cong K_1(\mathrm{S}^2\mathcal{A}) \text{ with isomorphism given by } \beta_{\mathrm{S}\mathcal{A}} \circ \Theta_{\mathcal{A}}$$

and

$$\beta_{\mathcal{B}} : K_0(\mathcal{B}) \to K_1(\mathrm{S}\mathcal{B})$$
$$\beta_{\mathcal{B}}([p]_0 - [1_n]_0) = [f_p]_1\,,\ f_p : [0,1] \to \mathcal{B}\,;\ t \mapsto \mathrm{e}^{2\pi i t p}$$
$$\Theta_{\mathcal{B}} : K_1(\mathcal{B}) \to K_0(\mathrm{S}\mathcal{B}) \text{ the index map for the SES:}$$
$$0 \to \mathrm{S}\mathcal{B} \hookrightarrow \mathrm{C}\,\mathcal{B} \overset{\mathrm{ev}_1}{\to} \mathcal{B} \to 0$$
$$\beta_{\mathcal{B}} \text{ and } \Theta_{\mathcal{B}} \text{ are natural}$$

For now forget about K_n for $n \geq 2$.

$$* \in \mathbb{Z} \text{ (grading)}$$
$$K_{*+1} = \begin{cases} K_0 & \text{if } * = 1 \\ K_1 & \text{if } * = 0 \end{cases}$$
$$\beta^*_{\mathcal{B}} = \begin{cases} \beta_{\mathcal{B}} & \text{if } * = 0 \\ \Theta_{\mathcal{B}} & \text{if } * = 1 \end{cases}$$

Theorem (Connes'-Thom Isomorphism)**.** *Let $(\mathcal{A}, \alpha, \mathbb{R})$ be a C*-dynamical system. There is a canonical isomorphism*

$$\phi^*_{\mathcal{A},\alpha} : K_*(\mathcal{A}) \to K_{*+1}(\mathcal{A} \rtimes_\alpha \mathbb{R})$$

Remark.

1) If $\alpha = \mathrm{id}$, then $\mathcal{A} \rtimes_{\mathrm{id}} \mathbb{R} \overset{\mathrm{FT}}{\cong} \underbrace{C_0(\mathbb{R}, \mathcal{A})}_{\cong C_0(]0,1[,\mathcal{A})} \cong \mathrm{S}\mathcal{A}$ so the above theorem tells us

$$K_*(\mathcal{A}) \overset{\phi^*_{\mathcal{A},\alpha}}{\cong} K_{*+1}(\mathrm{S}\mathcal{A}) \overset{\phi^{*+1}_{\mathrm{S}\mathcal{A},\mathrm{id}}}{\cong} K_*(\mathrm{S}^2\mathcal{A})$$

[20] G. Elliott, T. Natsume, R. Nest, *The Heisenberg Group and K-Theory*, K-Theory, 7, 409-428 (1993)

2) *Canonical* means: $\phi^{0,1}$ are the only natural isomorphisms which exist up to a choice of a factor of -1. We can fix the factor: choose $K_0(\mathbb{C}) \cong \mathbb{Z} \overset{\text{choice}}{\cong} \mathbb{Z} = K_1(\mathrm{S}\mathbb{C})$

The proof (of Bott periodicity) uses C*-fields. A C*-field over a locally compact Hausdorff space $I(= [0,1])$ is a triple

$$\Big(\underset{\text{C*-alg}}{\mathcal{A}} \,,\ \underset{\substack{\text{family of} \\ \text{C*-algs}}}{(\mathcal{A}_h)_{h\in I}} \,,\ \underset{\mathcal{A} \xrightarrow[\text{morphism}]{\phi_h} \mathcal{A}_h}{(\phi_h)_{h\in I}} \Big)$$

$$\mathcal{A} \text{ is an algebra of sections}\,; \quad \mathcal{A} \subset \prod_{h\in I} \mathcal{A}_h$$

all satisfying certain axioms:

1) $\forall\, a \in \mathcal{A}\,,\ \|a\|_{\mathcal{A}} = \sup_h \|\phi_h(a)\|_{\mathcal{A}_h}$
2) $\forall\, a \in \mathcal{A}\,,\ h \mapsto \|\phi_h(a)\|$ is continuous
3) $\mathcal{A}$ is a left $C_0(I)$-module

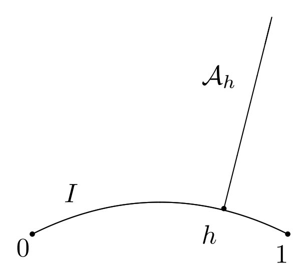


Figure 9: C*-field on $I = [0,1]$

$\mathcal{A}$ is trivial if $\mathcal{A} \cong C_0(I, \mathcal{B})$, where $\mathcal{B} \cong \mathcal{A}_h\ \forall\, h$. We need $I = [0,1]$ and $\mathcal{A}$ trivial except at 0; hence $\mathcal{A}|_{I\setminus\{0\}} \cong C_0(I \setminus \{0\}, \mathcal{A}_1)$, where $\mathcal{A}_1$ is a convenient choice from the $\mathcal{A}_h$ for $h > 0$.

Main example: $(\mathcal{B}, \alpha, \mathbb{R})$ C*-dynamical system

$$\underset{\text{new action}}{\alpha^h : \mathbb{R} \to \mathrm{Aut}(\mathcal{B})}\,,\ \alpha^h_\xi(b) := \alpha_{h\xi}(b)\,,\ h \in \mathbb{R}\,;\ \alpha^0 = \mathrm{id}\,,\ \alpha^1 = \alpha$$

$$\forall\, h > 0\,,\ \mathcal{B} \rtimes_{\alpha^h} \mathbb{R} \cong \mathcal{B} \rtimes_\alpha \mathbb{R}$$
$$f : \mathbb{R} \to \mathcal{B} \mapsto f^h : \mathbb{R} \to \mathcal{B}$$
$$f^h(t) = h^{1/2} f(ht)$$

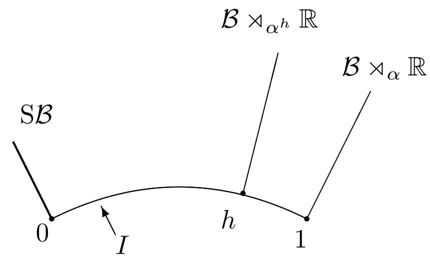


Figure 10: C*-field for a C*-dynamical system $(\mathcal{B}, \alpha, \mathbb{R})$

Theorem (Rieffel)**.** *If* $\mathcal{A} = C(I, \mathcal{B}) \rtimes_\tau \mathbb{R}$*, where* $(\tau_\xi f)(h) = \alpha_\xi^h(f(h))$*,* $f \in C(I, \mathcal{B})$*, and if* $\mathcal{A}_h = \mathcal{B} \rtimes_{\alpha^h} \mathbb{R}$*,* $\phi_h = \mathrm{ev}_h$*, then*

$$(\mathcal{A}, (\mathcal{A}_h, \phi_h)_{h\in I})$$

is a C-field, trivial except at 0.*

Lemma 1 (exercise)**.** *If* $(\mathcal{A}, (\mathcal{A}_h, \phi_h)_{h\in I})$ *is a C*-field, trivial except at* 0*, then so are*

unitization: $(\mathcal{A}^+, \mathcal{A}_h^+, \phi_h^+)$

stabilization: $(\mathsf{M}_n(\mathcal{A}), \mathsf{M}_n(\mathcal{A}_h), \mathsf{M}_n(\phi_h))$

suspension: $(\mathrm{S}\mathcal{A}, \mathrm{S}\mathcal{A}_h, \mathrm{S}\phi_h)$

Theorem. *If* $(\mathcal{A}, \mathcal{A}_h, \phi_h)_{h\in I=[0,1]}$ *is a C*-field, trivial away from* 0*, then there exists a unique map* $\mu_* : K_*(\mathcal{A}_0) \to K_*(\mathcal{A}_1)$ *such that*

1) μ_* *is natural;*

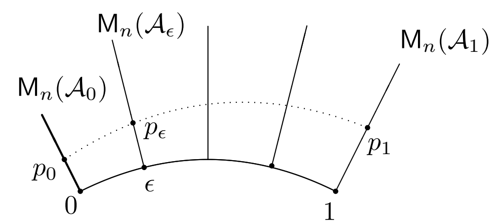
2) $\mu_* = \mathrm{id}$ *if* $\mathcal{A}$ *is a trivial field.*

Natural: given $(\mathcal{B}, \mathcal{B}_h, \phi_h)_{h\in I=[0,1]}$ and $f : \mathcal{A} \to \mathcal{B}$, $f_h : \mathcal{A}_h \to \mathcal{B}_h$, then the following diagram commutes:

$$
\begin{tikzcd}
K_*(\mathcal{A}_0) \arrow[r, "\mu_*"] \arrow[d, "K_*(f_0)"'] & K_*(\mathcal{A}_1) \arrow[d, "K_*(f_1)"] \\
K_*(\mathcal{B}_0) \arrow[r, "\mu_*"] & K_*(\mathcal{B}_1)
\end{tikzcd}
$$

Figure 11: Naturality of μ_*

Proof. The idea: $p \in \mathsf{M}_n(\mathcal{A})\,,\ p_0 = \phi_0(p)\,,\ p_1 = \phi_1(p) \rightsquigarrow \mu_*[p_0]_0 = [p_1]_0$



i) If $p_0 \in \mathsf{M}_n(\mathcal{A}_0)$ and $p, p' \in \mathsf{M}_n(\mathcal{A})$ such that $\phi_0(p) = \phi_0(p') = p_0$, we need to show that $[\phi_1(p)]_0 = [\phi_1(p')]_0$. By axiom 2), $h \mapsto \|\phi_h(p) - \phi_h(p')\|$ is continuous, so

$$\exists\, a > 0 : \forall\, h \leq a : \|\phi_h(p) - \phi_h(p')\| < 1$$

But $\phi_h(p)$ and $\phi_h(p')$ are projections, so $[\phi_h(p)]_0 = [\phi_h(p')]_0$.

If $p_0 \sim p_0' \in \mathsf{M}_n(\mathcal{A}_0)$, then $\exists\, \underset{\text{cont.}}{(q_t)_{t\in[0,1]}} \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A}_0)$ such that $q_0 = p_0$ and $q_1 = p_0'$. So

$$\exists\, (i_j)_j \,:\, 0 = i_0 < i_1 < \cdots < i_m = 1 \text{ and } \|q_j - q_{j+1}\| < \frac{1}{2}$$

Now the same argument.

So μ_* is well-defined for $* = 0$. For $* = 1$, same sort of thing using unitaries (or use $K_1 = K_0 \circ \mathrm{S}$).

ii) Group homomorphism; easy.

iii) If $\mathcal{A}$ is trivial then any section $p \in \mathsf{M}_n(\mathcal{A})$ defines a homotopy between p_0 and p_1, so $\mu_0 = \mathrm{id}$.

iv) Naturality of μ_* is an obvious diagram chase (see Fig. 11).

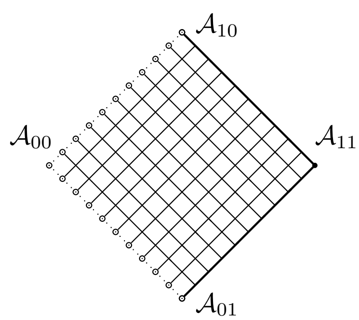
$\square$

Lemma 2. *μ_* is natural w.r.t β^*:*

$$\begin{tikzcd}
K_*(\mathcal{A}_0) \arrow[r, "\mu_*"] \arrow[d, "\beta^*_{\mathcal{A}_0}"'] \arrow[dr, phantom, "\textit{commutes}"] & K_*(\mathcal{A}_1) \arrow[d, "\beta^*_{\mathcal{A}_1}"] \\
K_{*+1}(\mathrm{S}\mathcal{A}_0) \arrow[r, "\mu_{*+1}"] & K_{*+1}(\mathrm{S}\mathcal{A}_1)
\end{tikzcd}$$

Proof. Exercise. $\square$

Lemma 3. *If $(\mathcal{A}, (\mathcal{A}_{(h,k)}, \phi_{(h,k)})_{(h,k)\in I\times I})$ is a C*-field over the square $I \times I$ which is trivial on $I \times I$ away from the edges $I \times \{0\}$ and $\{0\} \times I$, and is also trivial on $I \times \{0\}$ away from $(0,0)$, and again trivial on $\{0\} \times I$ away from $(0,0)$, then there is a commuting square of μ_* maps:*



$$\Rightarrow \quad \begin{tikzcd}
 & K_*(\mathcal{A}_{10}) \arrow[dr, "\mu_*"] & \\
K_*(\mathcal{A}_{00}) \arrow[ur, "\mu_*"] \arrow[dr, "\mu_*"'] \arrow[rr, phantom, "\textit{commutes}"] & & K_*(\mathcal{A}_{11}) \\
 & K_*(\mathcal{A}_{01}) \arrow[ur, "\mu_*"'] &
\end{tikzcd}$$

Proof. Exercise. $\square$

Theorem (Next lecture)**.** *$\mu_0 : K_0(\mathcal{A}_0) \to K_0(\mathcal{A}_1)$, where $\mathcal{A}$ is the Heisenberg group C*-algebra (see Fig. 12), is an isomorphism:*

$$[b]_0 - [1]_0 \mapsto \text{ class of rank 1 projections}$$

.

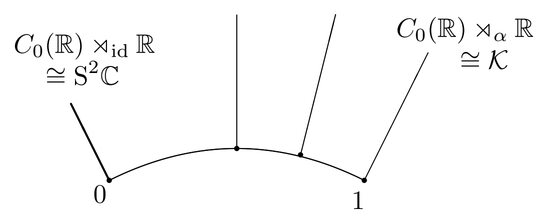


Figure 12: C*-field given by the Heisenberg group algebra

33 (20 April 2009)

Proof of the Bott theorem, part 2.

Key theorem. C*-field given by the Heisenberg group algebra (restricted to I):

$$\begin{aligned} C^*(H) &= C_0(\mathbb{R}\times\mathbb{R}) \rtimes_\tau \mathbb{R}\,,\ (\tau_\xi(f))(h,q) = f(h,q-h) \\ \mathcal{A} &= C_0(I\times\mathbb{R}) \rtimes_\tau \mathbb{R}\,,\ I=[0,1] \\ \mathcal{A}_h &= C_0(\{h\}\times\mathbb{R}) \rtimes_{\alpha^h} \mathbb{R} \\ \phi_h &= \mathrm{ev}_h \end{aligned}$$

This is trivial away from 0. Then (see Fig 13)

$$\mu_0 : \underbrace{K_0(\mathcal{A}_0)}_{\cong\mathbb{Z}} \to \underbrace{K_0(\mathcal{A}_1)}_{\cong\mathbb{Z}}$$
$$\mu_0([b]_0 - [1]_0) = [p]_0\,,\ \text{where } p \text{ is a rank 1 projection in } \mathcal{K}$$

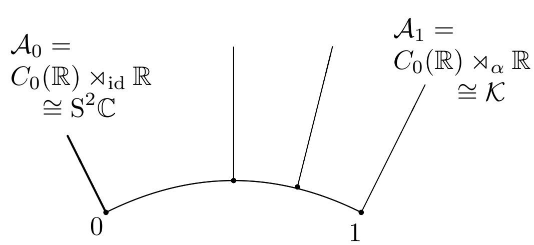


Figure 13: C*-field given by the Heisenberg group algebra

Recall from the previous lecture: for a C*-dynamical system $(\mathcal{A},\alpha,\mathbb{R})$,

$$\underbrace{\alpha^h : \mathbb{R}\to \mathrm{Aut}(\mathcal{A})}_{\text{new action}}\,,\ \alpha^h_\xi(b) := \alpha_{h\xi}(b)\,,\ h\in\mathbb{R}\,;\ \alpha^0 = \mathrm{id}\,,\ \alpha^1 = \alpha$$
$$\forall\, h>0\,,\ \mathcal{A}\rtimes_{\alpha^h}\mathbb{R} \cong \mathcal{A}\rtimes_\alpha \mathbb{R}$$

Theorem (Rieffel)**.** *If* $(\tau_\xi f)(h) = \alpha_\xi^h(f(h))$, $f \in C(I, \mathcal{B})$, *and if* $\phi_h = \mathrm{ev}_h$, *then*

$$(C(I, \mathcal{A}) \rtimes_\tau \mathbb{R},\, \mathcal{A} \rtimes_{\alpha^h} \mathbb{R},\, \phi_h)_{h \in I}$$

is a C-field, trivial except at 0 (see Fig. 14).*

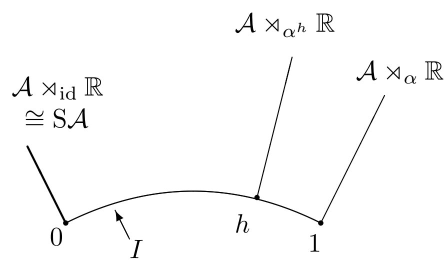


Figure 14: C*-field for a C*-dynamical system $(\mathcal{A}, \alpha, \mathbb{R})$

The map $\mu_* : K_*(\mathrm{S}\mathcal{A}) \to K_*(\mathcal{A} \rtimes_\alpha \mathbb{R})$ appears in the proof. For the Heisenberg field, $\mathcal{A} = C_0(\mathbb{R})$, $\alpha =$ translation.

Recall $\mathcal{A} \rtimes_\alpha \mathbb{R} \rtimes_{\hat{\alpha}} \mathbb{R} \cong \mathcal{A} \otimes \mathcal{K}(L^2(\mathbb{R}))$; Takai duality (see the 11 March lecture).

Big commuting diagram (utilizes lemmas from the previous lecture):

$$\begin{tikzcd}
 & & K_{*+1}(\mathcal{A} \rtimes_\alpha \mathbb{R}) \arrow[dr, "\beta^{*+1}_{\mathcal{A} \rtimes_\alpha \mathbb{R}}"] \arrow[drr, "\phi^{*+1}_{\mathcal{A} \rtimes_{\mathrm{id}} \mathbb{R}}"] & & \\
K_*(\mathcal{A}) \arrow[urr, "\phi^*_{\mathcal{A},\alpha}"] \arrow[r, "\beta^*_{\mathcal{A}}"] & K_{*+1}(\mathcal{A} \rtimes_{\alpha^0} \mathbb{R}) \arrow[ur, "\mu_{*+1}"] \arrow[dr, "\beta^{*+1}_{\mathrm{S}\mathcal{A}}"'] & \mathtt{II} & K_*(\mathcal{A} \rtimes_\alpha \mathbb{R} \rtimes_{\widehat{\alpha^0}} \mathbb{R}) \arrow[r, "\mu_*"] \arrow[d, phantom, "\mathtt{III}"] & K_*(\mathcal{A} \rtimes_\alpha \mathbb{R} \rtimes_{\hat{\alpha}} \mathbb{R}) \\
 & & K_*(\mathrm{S}^2\mathcal{A}) \arrow[ur, "\mu_*"] \arrow[r, "\mu_*"] & K_*(\mathrm{S}\mathcal{A} \rtimes_{\widehat{\alpha^0}} \mathbb{R}) \arrow[ur, "\mu_*"] &
\end{tikzcd}$$

Figure 15: Commuting diagram: `II` by lemma 2 ; `III` by lemma 3

We want to prove that $\phi^*_{\mathcal{B}}$ are isomorphisms for all C*-algebras $\mathcal{B}$. This will be the case if the compositions $\phi^{*+1}_{\mathcal{A} \rtimes_\alpha \mathbb{R}} \circ \phi^*_{\mathcal{A}} = \mathrm{id}$. In Fig. 15 this is the composition along the top of the diagram which may be replaced by the composition along the bottom. But first Fig. 15 will be modified by means of standard identifications and Takai duality:

2$^{\mathrm{nd}}$***K*****-group, middle row:**

$$K_{*+1}(\mathcal{A} \rtimes_{\alpha^0} \mathbb{R}) \cong K_{*+1}(\mathrm{S}\mathcal{A}) \text{ because } \mathcal{A} \rtimes_{\alpha^0} \mathbb{R} \cong \mathrm{S}\mathcal{A}$$

Last K-group, bottom row:

$$K_*(\mathrm{S}\mathcal{A} \rtimes_{\widehat{\alpha^0}} \mathbb{R}) \cong K_*(\mathcal{A}) \text{ because } \mathrm{S}\mathcal{A} \underset{(\text{or } \rtimes_{\widehat{\mathrm{id}}})}{\rtimes_{\widehat{\alpha^0}}} \mathbb{R} = (\mathcal{A} \rtimes_{\alpha^0} \mathbb{R}) \underset{(\text{or } \rtimes_{\widehat{\mathrm{id}}})}{\rtimes_{\widehat{\alpha}}} \mathbb{R}$$

$$\text{and then } \mathcal{A} \rtimes_{\alpha^0} \mathbb{R} \overset{\mathrm{FT}}{\cong} C_0(\mathbb{R}, \mathcal{A}) \Rightarrow (\mathcal{A} \rtimes_{\alpha^0} \mathbb{R}) \rtimes_{\widehat{\mathrm{id}}} \mathbb{R} \cong C_0(\mathbb{R}, \mathcal{A}) \rtimes_{\mathrm{transl}} \mathbb{R}$$

$$\cong \mathcal{A} \otimes \underbrace{C_0(\mathbb{R}) \rtimes_{\mathrm{transl}} \mathbb{R}}_{\mathcal{K}} \cong \mathcal{A} \otimes \mathcal{K} \text{ (Takai duality)}$$

$$\text{and, finally, } K_*(\mathcal{A} \otimes \mathcal{K}) \cong K_*(\mathcal{A})$$

NE arrow, bottom row: $\mu_* = \mathrm{id}$ presumably by using naturality to track generators.

$$\begin{tikzcd}
 & & K_{*+1}(\mathcal{A} \rtimes_\alpha \mathbb{R}) \arrow[dr, "\beta^{*+1}_{\mathcal{A}\rtimes_\alpha\mathbb{R}}"] \arrow[drrr, "\phi^{*+1}_{\mathcal{A}\rtimes_{\mathrm{id}}\mathbb{R}}"] & & & \\
K_*(\mathcal{A}) \arrow[urr, "\phi^*_{\mathcal{A},\alpha}"] \arrow[r, "\beta^*_{\mathcal{A}}"] & K_{*+1}(\mathrm{S}\mathcal{A}) \arrow[ur, "\mu_{*+1}"] \arrow[dr, "\beta^{*+1}_{\mathrm{S}\mathcal{A}}"'] & \mathtt{II} & K_*(\mathcal{A} \rtimes_\alpha \mathbb{R} \rtimes_{\widehat{\alpha^0}} \mathbb{R}) \arrow[rr, "\mu_*"] & & K_*(\mathcal{A}) \\
 & & K_*(\mathrm{S}^2\mathcal{A}) \arrow[ur, "\mu_*"] \arrow[rr, "\mu_*"] & \mathtt{III} & K_*(\mathcal{A}) \arrow[ur, "\mu_*=\mathrm{id}"'] &
\end{tikzcd}$$

Figure 16: Commuting diagram rewritten

We are left to show that

$$K_*(\mathcal{A}) \xrightarrow{\beta^*_{\mathcal{A}}} K_{*+1}(\mathrm{S}\mathcal{A}) \xrightarrow{\beta^{*+1}_{\mathrm{S}\mathcal{A}}} K_*(\mathrm{S}^2\mathcal{A}) \xrightarrow{\mu_*} K_*(\mathcal{A})$$

is the identity.

$\underline{* = 0}$: ($\mathcal{A}$ unital) $p \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A})$ (we may suppose $n = 1$ on replacing $\mathcal{A}$ by $\mathsf{M}_n(\mathcal{A})$). The map $i_p : \mathbb{C} \to \mathcal{A}$ defined by $1 \mapsto p$ leads to

$$\begin{tikzcd}
K_0(\mathcal{A}) \arrow[r, "\beta^0_{\mathcal{A}}"] & K_1(\mathrm{S}\mathcal{A}) \arrow[r, "\beta^1_{\mathcal{A}}"] & K_0(\mathrm{S}^2\,\mathcal{A}) \arrow[r, "\mu_0"] & \underset{=K_0(\mathcal{A})}{K_0(\mathrm{S}\mathcal{A}) \rtimes_\alpha \mathbb{R})} \\
K_0(\mathbb{C}) \arrow[r, "\beta^0_{\mathbb{C}}"] \arrow[u, "K_0(i_p)"] & K_1(\mathrm{S}\mathbb{C}) \arrow[r, "\beta^1_{\mathbb{C}}"] \arrow[u, "K_1(\mathrm{S}\, i_p)"] & K_0(\mathrm{S}^2\,\mathbb{C}) \arrow[r, "\mu_0"] \arrow[u, "K_0(\mathrm{S}^2\, i_p)"] & \underset{=K_0(\mathbb{C})}{K_0(C_0(\mathbb{R}) \rtimes_{\mathrm{transl}} \mathbb{R})} \arrow[u, "K_0(i_p)"']
\end{tikzcd}$$

$$[1]_0 \overset{\beta^0_{\mathcal{A}}}{\longmapsto} [(t \longmapsto \mathrm{e}^{2\pi i t 1})]_1 \overset{\beta^1_{\mathcal{A}}}{\longmapsto} [b]_0 - [1]_0 \overset{\mu_0}{\longmapsto} [q]_0$$

where μ_0 in the bottom row is the μ_0 of the key theorem, q is a rank 1 projection, so that $[q]_0 = [1]_0$, and by naturality, the composition in the top row carries $[p]_0$ to $[p]_0$:

$$\begin{tikzcd}
{[p]_0} \arrow[r, mapsto] & \cdots \arrow[r, mapsto] & {[p]_0} \\
{[1]_0} \arrow[r, mapsto] \arrow[u, "K_0(i_p)"] & \cdots \arrow[r, mapsto] & {[1]_0} \arrow[u, "K_0(i_p)"']
\end{tikzcd}$$

Hence the above composition is the identity.

34 (22 April 2009)

(Notes need further attention)

Takai duality: $\mathcal{A} \rtimes_\alpha \mathbb{R} \rtimes_{\hat\alpha} \mathbb{R} \cong \mathcal{A} \otimes \mathcal{K}$, and so $K_*(\mathcal{A} \rtimes_\alpha \mathbb{R} \rtimes_{\hat\alpha} \mathbb{R}) \cong K_*(\mathcal{A})$

If ϕ is any automorphism of $\mathcal{K}$, then $K_0(\phi) = \mathrm{id}$ (the eq. class of a projection is given by the rank which does not change).

Any automorphism of $\mathcal{B}(\mathcal{H})$ is inner:

$$\begin{gathered}\exists\, U^{\text{unitary}} \in \mathcal{B}(\mathcal{H}) \text{ s.th. } \phi = \mathrm{Ad}_U\,;\, \phi(A) = \mathrm{Ad}_U\, A = UAU^* \\ \Longrightarrow \text{if } A \text{ is trace-class, } \mathrm{Tr}(\phi(A)) = \mathrm{Tr}(A)\end{gathered}$$

Analogue:

$$\begin{aligned}
&(X,\phi) : \quad \mathbb{R}\text{-action on compact Hausdorff } X\\
\Longleftrightarrow &(C(X),\phi^*) : \quad \text{C*-dynamical system}\\
&(\mathbb{R}\times X, \mathrm{transl}\times\phi) : \quad \mathbb{R}\text{-action on } X\\
\Longleftrightarrow &(\underbrace{C_0(\mathbb{R}\times X)}_{\cong SC(X)}, \tau\times\phi^*) : \quad \rightsquigarrow \underbrace{SC(X)\rtimes_{\tau\times\phi^*}\mathbb{R}}_{\cong SC(X)\rtimes_{\tau\times\mathrm{id}}\mathbb{R}}\\
&\text{(undo the action by an automorphism)}\\
&(\mathbb{R}\times X, \mathrm{transl}\times\phi) \overset{\text{conj.}}{\cong} (\mathbb{R}\times X, \mathrm{transl}\times\mathrm{id})\\
&\qquad\qquad (t,x) \longmapsto (t,\phi_t(x))\\
&\qquad\quad \text{(that's why Takai duality works)}
\end{aligned}$$

Key Theorem.[21] If $(\mathcal{A},(\mathcal{A}_h)_h,(\phi_h)_h)$ is the Heisenberg field restricted to $I = [0,1]$:

$$\begin{aligned}
\mathcal{A} &= C_0(I\times\mathbb{R}) \rtimes_\tau \mathbb{R}\,,\ (\tau_\xi(f))(h,x) = f(h,x-h)\\
\mathcal{A}_h &= C_0(\mathbb{R}) \rtimes_{\alpha^h} \mathbb{R}\,,\ (\alpha^h_\xi(f))(x) = f(x-h\xi)\\
\phi_h &= \mathrm{ev}_h
\end{aligned}$$

then[22]

$$\mu_0 : K_0(\mathcal{A}_0) \to K_0(\mathcal{A}_1) \text{ maps } [b]_0 - [1]_0 \text{ to } [p]_0$$

for some rank 1 projection[23] p in $\mathcal{K}$.

Proof. Recall that we have the quantization map $Q : \mathcal{A}_0 \to \mathcal{A}$ which is a $*$-linear map (not an algebra morphism). Given that $b \in \mathrm{Proj}\,\mathsf{M}_2(\mathcal{A}_0^+)$, we could look

[21]This theorem is to be proved. See Lecture 33 (April 20th) for the original statement of this key theorem. In particular, see Fig. 13.

[22]Regarding b see Lecture 24 (March 27th), Example 2).

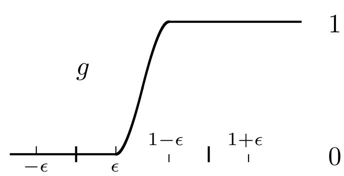
[23]The image of p in any irreducible representation of $\mathcal{K}$ is a projection of rank 1.

for a section $\tilde{b} \in \operatorname{Proj} \mathsf{M}_2(\mathcal{A})$ such that $\tilde{b}(0) = b$ and evaluate this at 1 to get $\tilde{b}(1)$; and then attempt to verify that $[\tilde{b}(1)]_0 = [1]_0 + [p]_0$. Apparently it is not possible to do it so simply.

Recall the definition of μ_0: given $q_0 \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A}_0^+)$, define $\mu_0([q_0]_0) = [q_1]_0$, $q_1 \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A}_1^+)$ provided $q_1 = p(1)$ and $q_0 = p(0)$ for some $p \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A}^+)$.

Question: given $q_0 \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A}_0^+)$ why is there $p \in \operatorname{Proj} \mathsf{M}_n(\mathcal{A}^+)$ such that $p(0) = q_0$? $\square$

There is always a self-adjoint $a \in \mathcal{A}$ such that $a(0) = p$. We want to make this into a projection. $\exists \epsilon > 0 \, : \, \forall h < \epsilon \ \operatorname{Spec}(a(h)) \subset \underbrace{B_\epsilon}_{\epsilon\text{-nbhd}}(\operatorname{Spec}(p))$. Let g be a continuous function of the form:



Then $g(a) \in \mathcal{A}$ and $\forall \, h < \epsilon$

$$\operatorname{Spec}(g(a(h))) = \{0,1\} \, (\Longleftrightarrow g(a(h)) \text{ is a projection})$$

Since the field is trivial beyond ϵ (on $[\epsilon, 1]$), $\exists$ projection-valued a; namely $g(a)$ [!]

However, such a straightforward approach does not work because we can't compute $g(a)$ unless we can diagonalize the Bott projection.

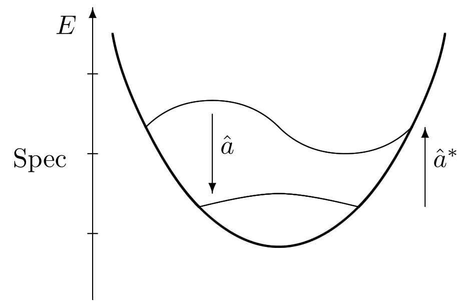
Idea: do it from the back. Find a nice projection in $\mathcal{A}_\hbar =$ obs. alg. assoc. to 1-d QM system with affiliated potential[24]

$$H = -\frac{\hbar^2}{2m}\, \partial_x^2 + V \text{ with purely discrete spectrum (non-degenerate)}$$

Example: The harmonic oscillator H (we all know its spectral theory).

$$\begin{aligned} \hat{p} &= \frac{\hbar}{i}\partial_x \, , \ \hat{q} = M_x \text{ (multiplication by } x) \\ \hat{a} &= \hat{p} - i\hat{q} \\ [\hat{a}, \hat{a}^*] &= \hbar \, \mathrm{id} \end{aligned}$$

[24]Lecture 5, Jan. 30th.



Π_h = projection onto the ground state of the harmonic oscillator = projection onto $\mathrm{Ker}\, a_h$.

Representation

$$\pi_1 : \mathcal{A}_h \to \mathcal{B}(L^2(\mathbb{R}))\,,\ \text{faithful, irreducible, } \mathrm{Im}\,\pi_1 = \mathcal{K}(L^2(\mathbb{R}))$$
$$\widetilde{\Pi}_h = \pi_1^{-1}(\Pi_h) \in \mathcal{A}_h$$

Naive idea: $(\widetilde{\Pi}_h)_h \in (\mathcal{A}_h)_h$ forms a section in $\mathcal{A}$?

It turns out that if $h \geq \epsilon$ then we have a section in $\mathcal{A}|_{[\epsilon,1]}\,, \forall\, \epsilon > 0$. There can't be norm convergence of $h \to 0$ because the spectrum becomes $\mathbb{R}^+$ (and absolutely continuous).

We need another idea. If $h = 0$, $\pi_1 : \mathcal{A}_h \to \mathcal{B}(L^2(\mathbb{R}))$ can't be faithful because $\mathcal{A}_0 = C_0(\mathbb{R}) \rtimes_{\mathrm{id}} \mathbb{R}$. Look into a representation $\pi_2 : \mathcal{A}_h \to \mathcal{B}(L^2(\mathbb{R}))$ which is faithful also for $h = 0$ but highly reducible.

Recall:[25] given a representation ρ of a C*-algebra $\mathcal{B}$ on $\mathcal{H}$ and an $\mathbb{R}$-action β on $\mathcal{B}$ we can induce a representation $\mathrm{Ind}[\rho]$ of $\mathcal{B} \rtimes_\beta \mathbb{R}$ on $L^2(\mathbb{R}, \mathcal{H})$. If ρ is faithful, so is $\mathrm{Ind}[\rho]$.

$$\mathcal{A}_h = C_0(\mathbb{R}) \rtimes_{\alpha^h} \mathbb{R} \overset{\mathrm{FT}}{\cong} \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\widehat{\mathrm{id}}^h} \mathbb{R} \quad (h \neq 0\,, \text{Takai})$$

so start with $\mathrm{id} : \mathbb{C} \to \mathbb{C}$; $\pi_2 = \mathrm{Ind}[\mathrm{Ind}[\mathrm{id}]]$

Lemma (homework)**.** *Representing*[26]

$$\widetilde{p} = \frac{\hbar}{i}\partial_x\,,\ \widetilde{q} = M_x - \partial_y\ (\textit{still}\ [\widetilde{p}, \widetilde{q}] = -i\hbar\,\mathrm{id})$$

is the same as π_2.

[25] See Lecture 3, 26 January.
[26] Notation changed for consistency with the subsequent lecture (Lecture 35)

Trick. Let

$$P_{a_h} = \text{ graph projection of } a_h = \frac{\hbar}{i}\partial_x - \left(M_x - \frac{\hbar}{i}\partial_y\right)$$

In general if T is an (unbounded) operator on $\mathcal{H}$, $\operatorname{graph}(T) = \{\psi \oplus T\psi \mid \psi \in \operatorname{dom}(T)\}$, T closed $\Leftrightarrow$ $\operatorname{graph}(T)$ closed in $\mathcal{H}$, and P_T is the projection onto $\operatorname{graph}(T)$.

So (since $1 + a_h^* a_h$ is invertible by positive spectrum)

$$P_{a_h} = \begin{pmatrix} (1 + a_h^* a_h)^{-1} & (1 + a_h^* a_h)^{-1} a_h^* \\ a_h(1 + a_h^* a_h)^{-1} & a_h(1 + a_h^* a_h)^{-1} a_h^* \end{pmatrix}$$

$$(1 + a_h^* a_h)^{-1} = \int_0^\infty \mathrm{e}^{-t(1+a_h^* a_h)}\,\mathrm{d}\,t \ \ \text{(cf. Mehler kernel)}$$

Claim $(P_{a_h})_h \in \pi_2(\mathcal{A})$; also $(P_{\lambda a_h})_h \in \pi_2(\mathcal{A})\ \ \forall\, \lambda > 0\,.$

$$P_{a_0} = \underset{\text{Bott projection}}{\pi_2(b)}$$

$$\begin{aligned}\lim_{\substack{\lambda\to\infty \\ h>0}} P_{\lambda a_h} &= \text{ projection onto} \lim_{\lambda\to\infty} (\psi, \lambda a_h \psi) \\ &= \text{ projection onto} \operatorname{Ker} a_h \oplus \overline{\operatorname{Im} a_h} = \pi_2(\widetilde{\Pi}_h) \oplus 1\end{aligned}$$

which proves

$$\mu_0([b]_0 - [1]_0) = [\tilde{\pi}_1]_0$$

35 (24 April 2009)

(Notes need further attention)

Recall:

$$\begin{aligned}
\mathcal{A} &= \text{restriction of the Heisenberg field to } [0,1] \\
&= C_0(I \times \mathbb{R}) \rtimes_\tau \mathbb{R} \\
\mathcal{A}_h &= C_0(\mathbb{R}) \rtimes_{\alpha^h} \mathbb{R} \quad \text{(fibers)} \\
\pi_1 : \quad & \underset{\text{faithful, irreducible } (h>0)}{\mathcal{A}_h \to \mathcal{K}(L^2(\mathbb{R}))} \ni \widehat{\Pi}_h \text{ (ground state proj. of harmonic osc.)} \\
\widehat{\alpha}_h &= \widehat{p} - i\widehat{q}\,,\ \widehat{p} = \frac{h}{i}\partial_x\,,\ \widehat{q} = x \text{ (unbounded)} \\
\widehat{H} &= \widehat{a}_h^* \widehat{a}_h \text{ (??)}
\end{aligned}$$

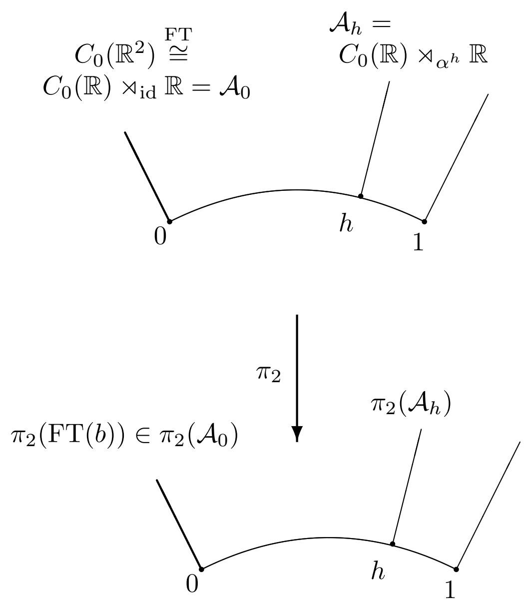


Figure 17: Heisenberg field on $[0,1]$ with projection π_2

In Fig. 17:

$$
\begin{aligned}
\pi_2 :& \mathcal{A}_h \to L^2(\mathbb{R}^2) \quad \text{(twice induced representation)}^{27} \\
\widetilde{\Pi}_h &= \pi_2 \pi_1^{-1}(\widehat{\Pi}_h) \\
\widetilde{a}_h &= \widetilde{p} - i\widetilde{q}\,,\ \ \widetilde{p} = \frac{h}{i}\partial_x\,,\ \ \widetilde{q} = x - \partial_y \\
[\widetilde{p}, \widetilde{q}] &= \frac{h}{i}\,\mathrm{id} \\
P_{\tilde{a}_h} &= \text{ proj. onto graph of } \widetilde{a}_h \\
\underset{h}{\sim} \lim_{\lambda\to\infty} P_{\lambda\tilde{a}_h} &= \begin{pmatrix} \widetilde{\Pi}_h & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Definition. Let a be a C^∞-function on $\mathrm{T}^*\mathbb{R}^n$. We define $(h > 0)$

$$
\begin{aligned}
\hat{a}_h :& \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \\
(\hat{a}_h\psi)(x) &= \frac{1}{2\pi}\int \mathrm{e}^{ix\cdot\xi} a(x, h\xi)\hat{\psi}(\xi)\,\mathrm{d}\,\xi \\
\hat{\psi}(\xi) &= \int \mathrm{e}^{-ix\cdot\xi}\psi(x)\,\mathrm{d}\,x
\end{aligned}
$$

This is well-defined if, e.g., a is of order m:

$$\forall\,\alpha \in \mathbb{N}^{2n} : |D^\alpha a(x,\xi)| \le C_\alpha(1 + \|(x,\xi)\|^2)^{\frac{m-|\alpha|_1}{2}} \ \text{ for } x, \xi \text{ large}$$

$\hat{a}_h$ *is a* Ψ*DO of order* m *with symbol* a.

$\hat{a}_h$ is *elliptic of order* m if

$$\exists\, C, R : |a(x,\xi)|^2 \ge C\|(x,\xi)\|^{2m} \ \text{ for } \|(x,\xi)\| \ge R$$

$\leadsto$ $\hat{a}_h$ is an unbounded Fredholm operator:

$$\mathrm{ind}\,(\hat{a}_h) = \dim\mathrm{Ker}\,\hat{a}_h - \dim\mathrm{Ker}\,\hat{a}_h^*$$

Now $m = 1$:

$$
\begin{aligned}
a(x,\xi) &= x + i\xi \\
\hat{a}_h &= x + h\partial_x = -i(\hat{p} - i\hat{q})
\end{aligned}
$$

$P_{\hat{a}_h}$ is the graph projection of $\hat{a}_h$, $\hat{a}_h$ is a ΨDO with symbol a , $b =$ the graph projection of a:

$$b = \begin{pmatrix} (1 + a^*a)^{-1} & (1 + a^*a)^{-1}a^* \\ a(1 + a^*a)^{-1} & a(1 + a^*a)^{-1}a^* \end{pmatrix}$$

[27] See Lecture 34, 22 April, for this and following.

$a \in C^\infty(T^*\mathbb{R}^n)$ (unbounded) acts by left multiplication on $L^2(T^*\mathbb{R}^n)$ $(n = 1)$, $b \in \mathsf{M}_2(C_0(T^*\mathbb{R}^n)^+)$.

$$\lim_{\lambda\to\infty} P_{\lambda\tilde{a}_h} = \begin{pmatrix} \text{proj}^{\text{n}} \text{ on } \operatorname{Ker}\tilde{a}_h & 0 \\ 0 & \text{proj}^{\text{n}} \text{ on } \operatorname{Im}\tilde{a}_h \end{pmatrix} = \underset{\tilde{\eta}_h = \text{ proj}^{\text{n}} \text{ onto } \operatorname{Ker}\tilde{a}_h^*}{\begin{pmatrix} \tilde{\pi}_h & 0 \\ 0 & 1-\tilde{\eta}_h \end{pmatrix}}$$

Theorem. *There is a section in the C^*-field*

$$C_0(I \times \mathbb{R}^n) \rtimes_\tau \mathbb{R}^n\,;\ (\tau_\xi f)(h,x) = f(h, x - h\xi)$$

linking $[b]_0$ *to* $[\tilde{\pi}_1]_0 - [\tilde{\eta}_1]_0$ *so*

$$\mu_0[b]_0 = [\hat{\pi}_1]_0 - [\hat{\eta}_1]_0$$

Recall $K_0(\mathcal{K}) \xrightarrow{\text{Tr}} \mathbb{Z}$, but $[\hat{\pi}_1]_0 - [\hat{\eta}_1]_0 \xrightarrow{\text{Tr}} \operatorname{ind}(\hat{a}_h)$. The analytic side of the index theorem is $\operatorname{ind}(\hat{a}_h) = \operatorname{Tr}_*([\pi_h]_0 - [\eta_h]_0)$ $(h > 0)$.

$n = 1$: full Heisenberg field $C_0(\mathbb{R}\times\mathbb{R}) \rtimes_\tau \mathbb{R}$. There is a canonical 3-trace (3 degrees of freedom). There is a canonical 2-trace on $C_0(\mathbb{R}\times\mathbb{R})^+ \cong C(\mathrm{S}^2)$; it is the one coming from the 2-cycle[28]

$$(\Omega(\mathrm{S}^2), d, \textstyle\int) = (C_0(\mathbb{R}\times\mathbb{R})^+ \otimes \underbrace{\bigwedge \mathbb{C}^2}_{\text{Grassmanian}}, d, \textstyle\int)$$

Using the τ-action:

$$\rightsquigarrow \text{3-cycle } (C_0(\mathbb{R}\times\mathbb{R}) \rtimes_\tau \mathbb{R} \otimes \bigwedge \mathbb{C}^3, \tilde{d}, \tilde{\textstyle\int}),\ \ \tilde{\textstyle\int} = \textstyle\int \circ \operatorname{ev}_1 \otimes i$$

Look back at the restriction $C_0(I\times\mathbb{R}) \rtimes_\tau \mathbb{R}$; try

$$(C_0(I\times\mathbb{R}) \rtimes_\tau \mathbb{R} \otimes \bigwedge \mathbb{C}^3, \tilde{d}, \tilde{\textstyle\int})$$

Is this a 3-cycle? All axioms for a 3-cycle satisfied except *closed*: $\tilde{\int}\tilde{d}\omega \neq 0$ (over a manifold with boundary; S^2 replaced by $I \times \mathrm{S}^1$). More next time.

[28] See Lecture 20, 9 March.

36 (27 April 2009)

(Notes need further attention)

Recall:[29] an n-cycle over $\mathcal{B}$ is a triple $(\Omega, d, \int)$:

- $\Omega = \bigoplus_{k\in\mathbb{Z}} \Omega_k$, graded algebra, $\Omega_k = 0$ for $k < 0$, $k > n$
- $\Omega_0 = \mathcal{B}$ and Ω_n is non-trivial
- d is a differential of degree 1 on Ω ($d^2 = 0$)
- $\int$ is a graded, closed trace:

graded: $\int \omega_1\omega_2 = (-1)^{\ell(n-\ell)} \int \omega_2\omega_1$, $\quad \omega_1 \in \Omega^\ell$, $\omega_2 \in \Omega^{n-\ell}$
closed: $\int d\omega = 0$ for $\omega \in \Omega^{n-1}$

This defines a cyclic n-cocycle η

$$\eta : \mathcal{B}^{n+1} \to \mathbb{C}\,;\ \ \eta(a_0, \ldots, a_n) = \int a_0 \,\mathrm{d}\, a_1 \cdots \mathrm{d}\, a_n$$

We call $(\Omega, d, \int)$ an *unbounded cycle* if d and $\int$ are only densely defined and $\exists$ a sub-algebra $\mathcal{B}' \subset \mathcal{B}$ (Banach algebra) such that η is fully defined on $\mathcal{B}'$, satisfying certain continuity conditions, as a consequence of which one can extend η to an intermediate algebra $\mathcal{B}' \subset \mathcal{B}'' \subset \mathcal{B}$ by continuity, and $\mathcal{B}''$ has the same K-theory as $\mathcal{B}$. Also, η can be extended to $\mathsf{M}_n(\mathcal{B})$ and to $\mathcal{B}^+$.

We define a functional $\underset{n \bmod 2}{K_n(\mathcal{B})} \to \mathbb{C}$

n even

$$\underset{p\in \mathrm{Proj}\, \mathsf{M}_k(\mathcal{B}^+)}{[p]_0 - [1]_0} \ \mapsto c_n \int \mathrm{Tr}((p-1)(\mathrm{d}\, p)^n) \text{ (independent of choice of } p)$$

$$\text{(for } k = 1 \text{ and } \mathcal{B} \text{ unital } \ \mapsto c_n \int p(\mathrm{d}\, p)^n \ )$$

$$c_n = \frac{1}{(2\pi i)^{n/2}} \frac{1}{\frac{n}{2}!}\,;\ c_0 = 1\,;\ c_2 = \frac{1}{2\pi i}$$

n odd

$$\underset{u\in \mathcal{U}\mathsf{M}_k(\mathcal{B}^+)}{[u]_1} \ \mapsto c_n \int \mathrm{Tr}\left((u^* - 1)\,\mathrm{d}\, u(\mathrm{d}\, u^* \,\mathrm{d}\, u)^{\frac{n-1}{2}}\right)$$

(independent of choice of u)

$$c_{2k+1} = \frac{1}{(2\pi i)^{k+1}} \frac{1}{2^{2k+1}} \frac{1}{(k+\frac{1}{2})(k-\frac{1}{2})\cdots(\frac{1}{2})}\,;\ c_1 = \frac{1}{2\pi i}$$

[29]See Lecture 19, 6 March.

<u>Ex.</u>: $\mathcal{B} = C(\mathrm{S}^1)$

$$(C(\mathrm{S}^1) \otimes \underbrace{\textstyle\bigwedge \mathbb{C}^1}_{=\mathbb{C}\oplus\mathbb{C}},\, d,\, \textstyle\int)$$

$$\textstyle\bigwedge \mathbb{C} = \bigwedge^0 \mathbb{C} \oplus \bigwedge^1 \mathbb{C} = \underset{1}{\mathbb{C}} \oplus \underset{e}{\mathbb{C}}$$

$$\text{on } C^1(\mathrm{S}^1)\,,\ df = \underbrace{f'}_{\text{derivative}} \otimes e$$

$$\textstyle\int f \otimes e = \int_{\mathrm{S}^1} f\,;\ \int \mathrm{d}\, f = \int f' = e\,(\text{closed})$$

In this 1-cycle over the dense sub-algebra of smooth elements $C^\infty(\mathrm{S}^1) = \mathcal{B}'$,

$$(\Omega(\mathrm{S}^1), d, \textstyle\int)$$

$$= (\text{differential forms on } \mathrm{S}^1, \text{exterior derivative}, \text{integration of 1-forms over } \mathrm{S}^1)$$

If $u \in \mathcal{U}\mathsf{M}_k C(\mathrm{S}^1)$, then

$$[u]_1 \mapsto \frac{1}{2\pi i} \int \mathrm{Tr}(u^* - 1)\,\mathrm{d}\,u \overset{\int \mathrm{Tr}\,\mathrm{d}\,u = 0}{=} \frac{1}{2\pi i} \int \mathrm{Tr}(u^*\,\mathrm{d}\,u)$$

if we choose a representative which is smooth. This is the winding number of $\det(u)$; for $k = 1$,

$$[u]_1 \to \frac{1}{2\pi i} \int_{\mathrm{S}^1} u^* u'$$

Definition. An n-chain is an "n-cycle with boundary": a quintuplet

$$(\Omega, d, \textstyle\int, \partial\Omega, r)$$

where

> $(\Omega, d, \int)$ satisfies all axioms of an n-cycle except closedness of the graded trace;
>
> $\partial\Omega$ is a graded algebra of top dimension $n-1$: $(\partial\Omega)_k = 0$ if $k < 0$ or $k > n-1$;
>
> $r : \Omega \to \partial\Omega$ is a surjective algebra morphism:
>
> i) $d(\mathrm{Ker}\, r) \subset \mathrm{Ker}\, r$;
> ii) $\int d\omega = 0$ if $\omega \in \mathrm{Ker}\, r$.

Definition. The boundary of an n-chain $(\Omega, d, \int, \partial\Omega, r)$ is $(\Omega, d', \int')$ where, for ω chosen arbitrarily in $r^{-1}(\omega')$, $\omega' \in \partial\Omega$,

> $d'\omega' = r d\omega$;
>
> $\int' \omega' = \int d\omega$

(Simple exercise: $(\Omega, d', \int')$ is an $(n-1)$-cycle over $(\partial\Omega)$.)

Ex.: Orientable n-manifold M with boundary ∂M.

$$(C(M) \otimes \bigwedge \mathbb{C}^n, d, \textstyle\int) \supset (\underset{\text{diff forms}}{\Omega(M)}, \underset{\text{ext deriv}}{d}, \underset{\text{int of forms}}{\textstyle\int})$$

$$\int_M d\omega \overset{\text{Stokes}}{=} \int_{\partial M} \omega$$

$$\partial M \overset{i}{\hookrightarrow} M$$

$$C(M) \xrightarrow{i^*} C(\partial M)$$

$$\Omega(M) \to \Omega(\partial M)$$

$$r = i^*,\ \partial\Omega(M) := \Omega(\partial M)$$

Proposition. *Let $(\Omega, d, \int, \partial\Omega, r)$ be an n-chain over $\mathcal{B}$ (Banach algebra), n odd. Let $p \in \operatorname{Proj} \mathsf{M}_k((\partial\Omega)_0^+)$, and suppose $\tilde{p}$ is a projection-valued lift in $\mathsf{M}_k(\mathcal{B}^+)$. Then the map*

$$[p]_0 - [1]_0 \mapsto c_{n-1} \textstyle\int' \operatorname{Tr}((p-1)(d'p)^{n-1})$$

is trivial. (Same statement for n even but $u \in \mathcal{U}\mathsf{M}_k((\partial\Omega)_0^+)$.)

Proof.

$$\begin{aligned}\int' \operatorname{Tr}(r(\tilde{p}) - 1)(d' r(\tilde{p}))^{n-1} &= \int' \operatorname{Tr}(r(\tilde{p}) - 1)(r(d\tilde{p}))^{n-1} \\ &= \int d \operatorname{Tr}(\tilde{p} - 1)(d\tilde{p})^{n-1} = \int \operatorname{Tr}(d\tilde{p})^n = 0\end{aligned}$$

because n is odd and $\int$ is graded. $\square$

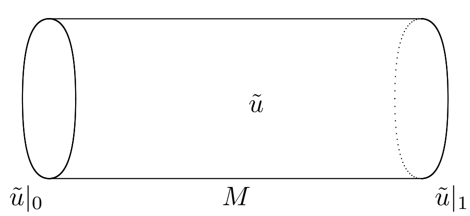


Figure 18: $\partial M = \mathrm{S}^1 \dot{\cup} \mathrm{S}^1$

Look at the chain

$$(\Omega(M), d, \textstyle\int, \Omega(\partial M), i^*) \rightsquigarrow$$

$$(\Omega(\partial M), d', \textstyle\int') = (\Omega(\mathrm{S}^1), d, \int_{\mathrm{S}^1}) \oplus (\Omega(\mathrm{S}^1), d, \underset{\uparrow \text{orientation}}{-}\!\int_{\mathrm{S}^1})$$

The pairing with $u = \tilde{u}_0 \oplus \tilde{u}_1$ is zero so the winding numbers are equal.

Aim: to construct a canonical chain for the Heisenberg field restricted to $[0, 1]$.

37 (29 April 2009)

(Notes need further attention)

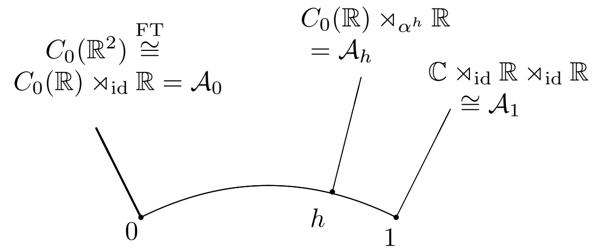


Figure 19: Heisenberg field on $[0,1]$

Q: Given a C*-field $\mathcal{A}$ over I, trivial away from 0, construct

$$n\text{-traces of } \mathcal{A}_0 \to n\text{-traces of } \mathcal{A}_1$$

(dual of the map[30] $\mu_* : K_*(\mathcal{A}_0) \to K_*(\mathcal{A}_1)$)

The idea is to construct an $(n+1)$-chain $(\Omega, d, \int, \partial\Omega, r)$ over the total field $\mathcal{A}$ such that the boundary of the chain is an n-cycle over $\mathcal{A}_0 \oplus \mathcal{A}_1$.

If (n even) $[p]_0 - [1]_0 \in K_0(\mathcal{A}_0)$, then lift p to $\tilde{p}$, a projection in $\mathcal{A}$; the Proposition from the previous lecture $\rightsquigarrow$ "character" of $(\Omega, d, \int)$ pairs trivially with $\tilde{p} \rightsquigarrow p = \tilde{p}(0) \oplus \tilde{p}(1)$ pairs trivially with the character of the boundary $(\partial\Omega, d', \int')$:

$$\begin{gathered}
\eta = \eta_0 \oplus \eta_1\,, \quad \eta([\tilde{p}(0)] \oplus [\tilde{p}(1)]) := \\
\int' \mathrm{Tr}\left[\left(\begin{pmatrix} \tilde{p}(0) & 0 \\ 0 & \tilde{p}(1) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\left(d'\begin{pmatrix} \tilde{p}(0) & 0 \\ 0 & \tilde{p}_1 \end{pmatrix}\right)^n\right] = 0 \\
\Rightarrow \int'_{\mathcal{A}_0} \mathrm{Tr}((\tilde{p}(0) - 1) d'(\tilde{p}(0))^n) = \int'_{\mathcal{A}_1} \mathrm{Tr}((\tilde{p}(1) - 1) d'(\tilde{p}(1))^n) \\
\Rightarrow \ \eta_0 \mapsto \eta_1 \text{ is dual to } \mu_*
\end{gathered}$$

Unrestricted Heisenberg field

$$C_0(\mathbb{R}^2) \rtimes_\tau \mathbb{R} = \mathbb{C} \rtimes_{\mathrm{id}} \underbrace{\mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R}}_{\tau \text{ acts on these}} \rtimes_{\mathrm{id}} \mathbb{R}$$

$$\tau_\xi(f)(h, x) = f(h, x - h\xi)$$

[30] See Lecture 32, 17 April.

Action by $\mathbb{R}$: extend n-cycle to $n+1$.

$$\begin{gathered} C_0(\mathbb{R}^2) = \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R} \\ (C_0(\mathbb{R}^2) \otimes \bigwedge \mathbb{C}^2,\, d,\, \textstyle\int_{\mathbb{R}^2} \otimes I)\,, \text{ where } \{e_1, e_2\} \text{ generates } \bigwedge \mathbb{C}^2\,, \\ f \in C_0(\mathbb{R}^2) \longmapsto df = \partial_1 f \otimes e_1 + \partial_2 f \otimes e_2\,, \text{ and} \\ i : \bigwedge^2 \mathbb{C}^2 \to \bigwedge^0 \mathbb{C}^2 \text{ is given by } e_1 \wedge e_2 \mapsto 1 \end{gathered}$$

τ_ξ does not commute with the derivative w.r.t. h, so $[\tau_\xi, h] \neq 0$. Simple trick: extend τ to an action $\tilde{\tau}$ on $C_0(\mathbb{R}^2) \otimes \bigwedge \mathbb{C}^2$ (graded algebra). τ_ξ is an automorphism of degree 0. For $f \in C_0(\mathbb{R}^2)^+$,

$$\begin{aligned} \tilde{\tau}_\xi(f \otimes 1) &= \tau_\xi(f) \otimes 1 \\ \tilde{\tau}_\xi(f \otimes e_1) &= \tau_\xi(f) \otimes e_1 \\ \tilde{\tau}_\xi(f \otimes e_2) &= \tau_\xi(f) \otimes (e_2 - \xi e_1) \end{aligned}$$

Now $[\tilde{\tau}_\xi, d] = 0$.

Proof.

$$\begin{aligned} (\tilde{\tau}_\xi d)(f \otimes 1) &= \tilde{\tau}_\xi(\partial_1 f \otimes e_1 + \partial_2 f \otimes e_2) \\ &= \tau_\xi(\partial_1 f) \otimes e_1 + \tau_\xi(\partial_2 f) \otimes (e_2 - \xi e_1)\,; \qquad (*) \\ (d\tilde{\tau}_\xi)(f \otimes 1) &= d(\tau_\xi(f) \otimes 1) \\ &= \partial_1 \tau_\xi(f) \otimes e_1 + \partial_2 \tau_\xi(f) \otimes e_2 \end{aligned}$$

From the definitions, if $(h, x) \in \mathbb{R}^2$,

$$\begin{aligned} \partial_1 \tau_\xi(f)(h,x) &= \frac{\partial}{\partial h} f(h, x - h\xi) = (\partial_1 f - \xi \partial_2 f)(h, x - h\xi) \\ &= \tau_\xi(\partial_1 f - \xi \partial_2 f)(h,x) \text{ and} \\ \partial_2 \tau_\xi(f)(h,x) &= \frac{\partial}{\partial x} f(h, x - h\xi) = (\partial_2 f)(h, x - h\xi) \\ &= \tau_\xi(\partial_2 f)(h,x)\,, \text{ so} \\ (d\tilde{\tau}_\xi)(f \otimes 1) &= \partial_1 \tau_\xi(f) \otimes e_1 + \partial_2 \tau_\xi(f) \otimes e_2 \\ &= \tau_\xi(\partial_1 f - \xi \partial_2 f) \otimes e_1 + \tau_\xi(\partial_2 f) \otimes e_2 \qquad (**) \end{aligned}$$

Comparing $(*)$ and $(**)$: $\tilde{\tau}_\xi$ and d commute. $\square$

So we get a 3-cycle

$$\left(\left(C_0(\mathbb{R}^2) \otimes \bigwedge \mathbb{C}^2\right) \rtimes_{\tilde{\tau}} \underbrace{(\mathbb{R} \hat{\otimes} \bigwedge \mathbb{C})}_{\text{graded } \otimes}\,,\ \tilde{d}\,,\ \textstyle\int_{\mathbb{R}^2} \circ\, \mathrm{ev}_0 \otimes i \right)$$

where

$$\begin{aligned}
i : \bigwedge^3 \mathbb{C}^3 &\to \mathbb{C} \\
f &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \tau_{t\cdot 1}(f) =: \delta f \\
\tilde{d} &= d + (-1)^{\mathrm{sign}}\delta
\end{aligned}$$

Claim: this is a 3-cycle.

Now restrict the 3-cycle to $I \times \mathbb{R} \subset \mathbb{R}^2$.

The "trace" $\int_{I\times\mathbb{R}} \circ\, \mathrm{ev}_0 \otimes i$ is graded but no longer closed. So we get a chain:[31]

$$\Big((C_0(I \times \mathbb{R}) \otimes \bigwedge \mathbb{C}^2) \rtimes_{\tilde{\tau}} (\mathbb{R} \hat{\otimes} \bigwedge \mathbb{C}),\ \tilde{d}\,,\ \textstyle\int_{I\times\mathbb{R}} \circ\, \mathrm{ev}_0 \otimes i,\ \partial\Omega\,,\ r \Big)$$

where

$$\partial\Omega = \left((\underset{\substack{\uparrow \\ C_0(\{0\},\mathbb{R}) \otimes C_0(\{1\},\mathbb{R})}}{C_0(\partial I, \mathbb{R})} \otimes \underset{\text{gen. by } e_2}{\bigwedge \mathbb{C}}) \rtimes_{\tilde{\tau}} (\mathbb{R} \hat{\otimes} \bigwedge \mathbb{C}),\ \tilde{d}'\,,\ \textstyle\int_{\partial I \times \mathbb{R}} \circ\, \mathrm{ev}_0 \otimes i \right)$$

$$\begin{aligned}
r &= \phi_0 \otimes \mathrm{id} \oplus \phi_1 \otimes \mathrm{id} \\
\tilde{d}' &= \partial_2 \otimes e_2 + (-1)^{\mathrm{sign}}\delta \\
i &: \bigwedge^2 \{e_2, e_3\} \to \mathbb{C}
\end{aligned}$$

Exercise.

$$\begin{gathered}
\tilde{d}\,\mathrm{Ker}\, r \subset \mathrm{Ker}\, r\,; \\
\textstyle\int \tilde{d}\omega = 0\,,\ \forall\, \omega \in \mathrm{Ker}\, r
\end{gathered}$$

[31] See Lecture 36, 27 April.

38 (1 May 2009)

Another Gap Labelling[32]

Rotation Numbers (Moser, 1-D).

Integrated Density of States (any dimension).

Now K_n-gap labelling (any dimension).

1-D differential operator $H = -\partial^2 + V$ (unit $\hbar^2/2m = 1$), where V is a differentiable function on $\mathbb{R}$.

Consider $E \in \mathbb{R}$,

$$(*) \qquad H\psi = E\psi\,,\ \psi : \mathbb{R} \to \mathbb{C} \quad (\psi \text{ non-zero})$$

Always solvable for $\psi \neq 0$ but the solution depends on E; this is where Moser started.

1) If E is an eigenvalue $\exists\, \psi \in L^2(\mathbb{R})$ solving $(*)$.

2) If $E \in \sigma(H)$ but not an eigenvalue:

   Simple if $\sigma(H) = \sigma_{\substack{\text{abs}\\\text{cont}}}(H) \cup \sigma_{\text{pure point}}(H)$; then ψ is bounded $\rightsquigarrow$ linear combinations of these to construct wave packets. This is difficult.

   If $E \in \sigma_{\text{sc}}(H)$ "ψ is critical". This was the devil until the 80's; then became fashionable.

   Remark: If $V = 0$, then $\sigma = \mathbb{R} = \sigma_{\text{ac}}$ then use Fourier transforms to construct wave packets. If $V \neq 0$ then?

3) If $E \notin \sigma(H)$ then $\exists!$ (up to multiplicative constants) solutions ψ_+, ψ_- which satisfy $\lim_{x \to \pm\infty} \psi_\pm(x) = 0$ and $x \to \mp\infty$ leads to exponential increase of $\psi_\pm$ (see Fig. 20). This excludes any possible interpretation of

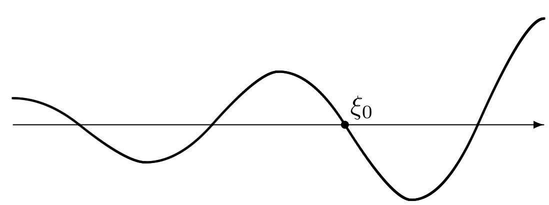


Figure 20: ψ_-

these particles. The only way is to restrict the system.

[32]Thanks to Carl Olimb for the notes.

Rotation Numbers: $E \notin \sigma(H)$; $\alpha(E) =$ "density of zeros" of $\psi_- = 2$ density of rotation of $\psi(x) + i\psi'(x)$. $\alpha(E)$ is constant on gaps.

What is the density in this context? We mean a sequence of intervals subdivided and take rotation numbers. So may depend on this sequence.

$$[a_n, b_n] \subset [a_{n+1}, b_{n+1}] \subset \cdots \to \mathbb{R}$$

The number of zeros in the interval is equal to the number of density states. Then

$$\alpha(E) = \mathrm{IDS}(E) = \langle 0\text{-}\underset{\ni\tau}{\text{Hull}} \,|\, [P(H)]_0 \rangle \text{ , } K_0\text{-label.}$$

Now $H_\xi = -\partial^2 + V_\xi$, $V_\xi(x) = V(x+\xi)$. $H_\xi \underset{\text{unitary}}{\sim} H$, so $\sigma(H_\xi) = \sigma(H)$. If $E^{(\text{fixed})} \notin \sigma(H)$, $H_\xi \psi_{\xi -} = E\psi_{\xi -}$ with $\psi_{\xi -}(x) \overset{x\to-\infty}{\longrightarrow} 0$. Then $\psi_{\xi -}(x) = \psi_-(x+\xi)$. Consider Fig. 20 again:

$$\begin{aligned} &\xi_0 \text{ is a zero of } \psi_- \\ \Leftrightarrow\ &0 \text{ is a zero of } \psi_{\xi_0 -} \\ \Leftrightarrow\ &E \text{ is an eigenvalue of } H_\xi \,|_{\mathbb{R}^-} \\ &\quad \text{with Dirichlet boundary condition at } 0 \end{aligned}$$

Call E a *Dirichlet value of* H_{ξ_0}.

Notation:

$$\hat{H}_{\xi_0} := H_{\xi_0} \,|_{\mathbb{R}^-} \ \text{ with Dirichlet boundary condition.}$$

Write $D_\xi(0) = D_\xi = \{$Dirichlet values of H_ξ in $\Delta\}$, where Δ is a gap in $\sigma(H_\xi)$ $= \sigma(H)$.

General remarks:

i) $\sigma_{\mathrm{ac}}(\hat{H}_\xi) \subset \sigma_{\mathrm{ac}}(H_\xi)$. So, $\sigma_{\mathrm{ac}}(\hat{H}_\xi) \subset \sigma(H) \cup \bigcup_{\text{gaps } \Delta} D_\xi(\Delta)$

ii) If $\alpha(\Delta) \neq 0$, then $|D_\xi(\Delta)| \leq 1$. "No second eigenvalue in the same gap."

Let $\mu(\xi) =$ EV of $\hat{H}_\xi$ in Δ.

$$(**) \qquad \{\text{zeros of } \psi_-\} = \{\xi \,|\, |D_\xi(\Delta)| = 1\}$$

Now vary ξ; we get curves as in Fig. 21.

Lemma 4. $\mu'(\xi) < 0$

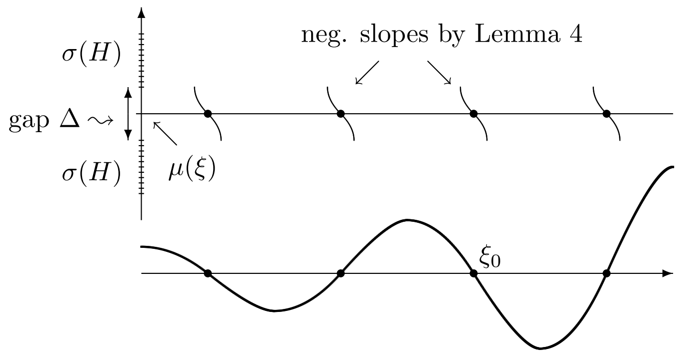


Figure 21: μ

Proof. $\mu(\xi) = \langle \hat\psi_\xi \,|\, \underbrace{\hat H_\xi}_{-\partial^2+V_\xi} \,|\, \hat\psi_\xi\rangle$

$$\frac{\partial\mu(\xi)}{\partial\xi} = \underbrace{\langle \hat\psi'_\xi \,|\, \hat H_\xi \,|\, \hat\psi_\xi\rangle + \langle \hat\psi_\xi \,|\, \hat H_\xi \,|\, \hat\psi'_\xi\rangle}_{\mu(\xi)(\langle\hat\psi_\xi,\hat\psi'_\xi\rangle+\langle\hat\psi'_\xi,\hat\psi_\xi\rangle)=\frac{\partial}{\partial\xi}|\hat\psi_\xi|^2=0} + \underbrace{\langle \hat\psi_\xi \,|\, V'_\xi \,|\, \hat\psi_\xi\rangle}_{\uparrow}$$

expectation of mechanical force of this wave funtion

$$\begin{aligned}
&= \int_{-\infty}^{0} \hat\psi_\xi V'_\xi \hat\psi_\xi \,\mathrm{d}\,x = -\int_{-\infty}^{0} \left(\hat\psi'_\xi V_\xi \hat\psi_\xi + \hat\psi_\xi + \hat\psi_\xi V_\xi \hat\psi'_\xi\right) \mathrm{d}\,x \\
&= \int_{-\infty}^{0} \left(\hat\psi'_\xi \partial^2 \hat\psi_\xi + \hat\psi_\xi \partial^2 \hat\psi'_\xi\right) \mathrm{d}\,x \\
&= -\int_{-\infty}^{0} \frac{\partial}{\partial x}(\hat\psi'_\xi \hat\psi'_\xi)\,\mathrm{d}\,x = -|\hat\psi'_\xi(0)|^2 < 0
\end{aligned}$$

□

Two important results: $-|\hat\psi'_\xi(0)|^2 < 0$ and the resultant force.

In our Fig. 21 the curves cannot cross since then we would have a degenerate eigenvalue. Now look at the "line" created by $\mu(\xi)$ on a circle.

$$\tilde\mu : \mathbb{R} \to \{z \in \mathbb{C} \,|\, |z| = 1\}\,, \text{ where } \tilde\mu(\xi) = \exp\left(2\pi i \,\frac{\mu(\xi) - E_0}{|\Delta|}\right)\,, \; E_0 = \inf\Delta$$

where $\mu(\xi) = E_0$ is no eigenvalue at ξ.

<u>Final Result</u>. Moser rotation number is the same as another rotation number.

$$(**) \qquad \begin{aligned}
\alpha(E) &= \text{density of zeros of } \psi_- \\
&= \text{density of intersections of } \mu \text{ with } E \\
&= \text{density of } \tilde\mu = \beta(E) = \beta(\Delta)
\end{aligned}$$

$$\begin{aligned}\beta(\Delta) &= \lim_{[a_n,b_n]\nearrow\mathbb{R}} \frac{\Delta}{|b_n-a_n|}\frac{\Delta}{2\pi i}\int_{a_n}^{b_n}\mu^*(\xi)\mu(\xi)\,\mathrm{d}\,\xi \\ &= \lim_{[a_n,b_n]\nearrow\mathbb{R}} \frac{\Delta}{|\Delta|}\int_{a_n}^{b_n}\mathrm{Tr}(-V'_\xi P_\Delta(\hat{H}_\xi))\,\mathrm{d}\,\xi\end{aligned}$$

where $P_\Delta(\hat{H})$ is the spectral projection to states in Δ (actually only one); so $P_\Delta(\hat{H}) = |\,\psi_\xi\rangle\langle\hat{\psi}_\xi\,|$. This can be interpreted as a pairing of a 1-trace with a K_1-class.

$$\begin{aligned}U_\xi &= \mathrm{e}^{2\pi i\frac{\hat{H}_\xi - E_0}{|\Delta|}}P_\Delta(\hat{H}_\xi) + P_\Delta(\hat{H}_\xi)^\perp \in \mathcal{K}(L^2(\mathbb{R}))^+ \\ U_\Delta &:= (\xi\mapsto U_\xi)\in C_0(\mathbb{R},\mathcal{K}(L^2(\mathbb{R}))^+) = \mathrm{S}\,\mathbb{C}\otimes\mathcal{K}^+ = C(\mathrm{S}^1)\otimes\mathcal{K}^+\end{aligned}$$

so $[U_\Delta]\in K_1(C_0(\mathbb{R}))$.

Now 1-trace = character of $(\underbrace{\Omega(\mathbb{R})}_{\text{DeRahm forms}}, d, \int_\mathbb{R})$. Then $\langle\eta\,|\,[U_\Delta]\rangle = \beta(\Delta)$.

To finish, $\alpha=\beta$ is an index theorem. Why?

$$[U_\Delta]_1 = \exp[\underset{\substack{\uparrow \\ \text{energy values } \leq E}}{P_{\leq E}}(H)]_0 \text{ and additional work}$$