Designs of Mixed Resolution for Process Robustness Studies

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A robust process is a process that is insensitive to changes in uncontrollable variables. In this article a class of designs that can be used to achieve a robust process is proposed. These new designs are similar in structure to classical central composite designs, but they are of *mixed resolution*. That is, the new designs are at least Resolution V among the signal factors and are at least Resolution III among the noise factors. A catalog of the new designs, known as composite mixed-resolution (CMR) designs, is included for the practitioner. A comparison of the sizes of robustness designs shows that many CMR designs are superior to or competitive with the corresponding Taguchi designs. The response surface models associated with these two classes of robustness designs are also compared. D efficiencies and G efficiencies of the CMR designs are included.

KEY WORDS: Composite design; D efficiency; G efficiency; Minimum aberration; Mixed-resolution design; Taguchi design.

1. EXPERIMENTAL DESIGNS FOR PROCESS ROBUSTNESS STUDIES

When the goal of an experiment is to acquire information about a manufacturing process, the experimenter is often concerned with the relationship between the levels of experimental design variables and a response of interest. When the experimental goal is specifically to acquire information that can lead to achieving a robust process, each design variable is commonly classified as a member of one of the following two sets of variables:

1. "Signal" variables are variables that are internal to the manufacturing process. Because they are internal to the process, the settings of the signal variables can be routinely controlled.
2. "Noise" variables are variables that are, in general, external to the process and are difficult or impossible to control during the manufacturing process.

Although noise variables are random during the manufacturing process, their levels can be controlled for experimental purposes.

For an experiment to be successful in achieving a robust process, it should provide information that can lead to a reduction in process variability attributable to changes in the noise-variable levels. This information will enable the experimenter to determine settings of the controllable process factors that will yield a product whose quality is acceptable to the consumer regardless of the levels of the noise variables.

The Taguchi system of experimental design for studying the robustness of a process (Taguchi 1991; Taguchi and Wu 1979) contains designs that are formed by "crossing" two designs, a signal-factor inner array and a noise-factor outer array. Taguchi referred to his designs as *orthogonal arrays.* Figure 1 illustrates a 36-point orthogonal array formed by crossing a four-factor three-level inner array of nine points with a three-factor two-level outer array of four points.

In the classical experimental-design literature (e.g., Kempthorne 1952; Cochran and Cox 1957), experimental designs were based on a single-factor array. Many authors, including Lucas (1989, 1994), Sacks, Welch, Mitchell, and Wynn (1989), Welch, Sacks, and Schiller (1990), Box and Jones (1990), Shoemaker, Tsui, and Wu (1991), and Myers, Khuri, and Vining (1992), suggested the use of single-factor arrays as alternatives to the use of crossed arrays. Discussions regarding designs that are single-factor arrays but have "mixed" design resolution were presented by Lucas (1989, 1994). Box and Jones (1990) also discussed designs of mixed resolution. In this article we further develop and evaluate properties of mixed-resolution designs.

2. MIXED DESIGN RESOLUTION: AN ALTERNATIVE DESIGN APPROACH

Researchers should be aware that, in general, estimation of the complete set of interactions among the experimental variables is not possible when a response surface analysis is applied to data collected from a Taguchi orthogonal array (Lucas 1989, 1994; Shoemaker et al. 1991). The most comprehensive model that can always be fit by crossing a three-level inner array with a two-level outer array is

$$y = \beta_0 + \sum_{i=1}^{C} \beta_i x_i + \sum_{i=1}^{C} \beta_{ii} x_i^2 + \sum_{j=1}^{U} \delta_j z_j + \sum_{i=1}^{C}\sum_{j=1}^{U} \delta_{ij} x_i z_j + \sum_{i=1}^{C}\sum_{j=1}^{U} \delta_{iij} x_i^2 z_j + \varepsilon, \quad (1)$$

where $\{x_i\colon\ i = 1,2,\ldots,C\}$ and $\{z_j\colon\ j = 1,2,\ldots,U\}$ are, respectively, the inner-array (controllable) process vari-

Inner				Outer				
				0	0	1	1	z_1
				0	1	0	1	z_2
x_1	x_2	x_3	x_4	0	1	1	0	z_3
0	0	0	0	y_{11}	y_{12}	y_{13}	y_{14}	R
0	1	1	2	y_{21}	y_{22}	y_{23}	y_{24}	E
0	2	2	1	y_{31}	y_{32}	y_{33}	y_{34}	S
1	0	1	1	y_{41}	y_{42}	y_{43}	y_{44}	P
1	1	2	0	y_{51}	y_{52}	y_{53}	y_{54}	O
1	2	0	2	y_{61}	y_{62}	y_{63}	y_{64}	N
2	0	2	2	y_{71}	y_{72}	y_{73}	y_{74}	S
2	1	0	1	y_{81}	y_{82}	y_{83}	y_{84}	E
2	2	1	0	y_{91}	y_{92}	y_{93}	y_{94}	S

Figure 1. A 36-Point Crossed-Array Design.

ables and the outer-array (uncontrollable) variables. Lucas (1989, 1994) called the model in (1) the *propagation-of-error model* because the recommended Taguchi two-stage analysis can be reproduced by applying the propagation-of-error formula (Deming 1946) to the model. It may be possible to include in the model additional terms that correspond to estimable effects. For example, the model may include some $x_i x_j$ or $z_k z_l$ interaction terms. Inclusion of such terms, however, depends on the alias structures of the inner and outer arrays. For other designs, such as the seven-factor design in Figure 1, inclusion of additional $x_i x_j$ or $z_k z_l$ interaction terms will not be considered due to the nonestimability of the corresponding effects.

Because inner-array (x_i) variables are controllable process variables, designs that do not allow the estimation of the $x_i x_j$ interaction effects can deny the experimenter valuable process information, especially when the goal is process optimization. In their reanalysis of Taguchi's Wheatstone Bridge circuit example, Box and Fung (1986) showed that a marginal means analysis could ignore a significant interaction and lead to suboptimal factor settings. When an orthogonal array does not allow estimation of the full set of interactions among the controllable process variables, process adjustments cannot be made to account for any of the nonestimable but influential interactions.

We now propose new robustness designs that will allow estimation of interaction effects among the controllable process variables. We begin by defining a *mixed-resolution* (MR) 2^{K-P} design to be any 2^{K-P} fractional factorial design satisfying the following three conditions:

1. Among the C signal factors, the design is at least Resolution V. That is, among the C signal factors, no main effects or two-factor interactions are aliased with any other main-effect or two-factor interaction.
2. Among the U noise factors, the design is at least Resolution III. That is, among the U noise factors, no main effects are aliased with any other main effect. Main effects, however, are aliased with two-factor interactions and two-factor interactions are aliased with each other.
3. None of the $C \times U$ signal-by-noise two-factor interactions are aliased with any main effect or any two-factor interaction.

Our new robustness designs, which are composite designs that are similar in structure to the central composite designs of Box and Wilson (1951), consist of the following:

1. $F = 2^{K-P}$ factorial points from an MR design with C signal and U noise factors and with levels coded as ± 1
2. $2C$ star points. For each signal factor the design includes two "star" points; that is, the signal factor is set at levels ± 1 and all other factors are set at mid-level 0. In some situations, it is worthwhile to replicate the star points for increased efficiency. See Section 5 for details. In general, the star-point signal-factor levels could take on any values $\pm\alpha$. In our research, we only consider the case in which the star-point distance is $\alpha = 1$. That is, the experimental design region is a hypercube (which is identical to the region used by Taguchi).
3. N_0 center points

Designs composed of the $N = F + 2C + N_0$ experimental points described in 1, 2, and 3 are called *composite mixed-resolution designs* or *CMR* designs. Every CMR design in this class of designs allows the estimation of the parameters in the CMR model:

$$y = \beta_0 + \sum_{i=1}^{C} \beta_i x_i + \sum_{i=1}^{C} \beta_{ii} x_i^2 + \sum_{i=1}^{C-1} \sum_{j=i+1}^{C} \beta_{ij} x_i x_j + \sum_{k=1}^{U} \delta_k z_k + \sum_{i=1}^{C} \sum_{k=1}^{U} \delta_{ik} x_i z_k + \varepsilon. \quad (2)$$

Many response surface designs for fitting second-order models (e.g., central composite designs and CMR designs) are based on a single-factor array. The z_i^2 and $z_i z_j$ noise terms that appear in full second-order response surface models, however, do not appear in (2). If the loss of information corresponding to these noise effects is not deemed important and the CMR model is acceptable to the experimenter, then implementation of a CMR design should be considered because it will require fewer experimental runs than the corresponding central composite design on $C + U$ factors. The alias structure of the chosen 2^{K-P} factorial design determines which effects are estimable and, hence, which of the $z_i z_j$ interaction terms could be incorporated into a modified version of Model (2). Moreover, the CMR model in (2) excludes the generally less important third-order $x_i^2 z_j$ terms in (1). For more information regarding the CMR model, see Lucas (1989, 1994).

Box and Jones (1990) also considered CMR designs as design alternatives to orthogonal arrays. To protect against bias due to any excluded $z_i z_j$ model terms, they considered only MR 2^{K-P} designs of Resolution IV among the noise variables when forming CMR designs. If the full second-order model is the true model, then these designs produce unbiased estimates of the coefficients of the terms involving only signal variables, thereby allowing for the minimization of the integrated squared error. Box and Jones showed that, in many cases, use of a CMR design will substantially reduce the design size while still allowing the experimenter to apply response surface techniques toward the achievement of a robust process.

In our construction of CMR designs, we allow the MR 2^{K-P} design to be Resolution III among the noise factors. We realize that implementation of these smaller designs could increase the effects of model bias. The majority of CMR designs are, however, formed from MR 2^{K-P} designs that are at least Resolution IV among the noise factors. For those limited cases when the design is Resolution III among the noise factors, we rely on the main-effects principle, which assumes that main effects tend to be the dominant effects in a quadratic response surface analysis. Therefore, when main effects are aliased with two-factor interactions among the noise factors, we expect that little information will be lost in a response surface analysis. In fact, if the CMR model is the true model, then a CMR design of Resolution III among the noise factors will produce unbiased estimates of the coefficients of the terms involving only signal factors.

If a CMR design is Resolution III among the noise factors, then it requires half the number of factorial points required by a CMR design of Resolution IV among the noise factors. We believe that design size efficiency is a common and practical concern for experimenters. Therefore, the possibility of some estimation bias to achieve the smaller design size may be an acceptable risk. If the true model contains higher-order terms, such as cubic terms, then even a CMR design of Resolution IV among the noise factors will yield biased estimates.

3. SMALL MIXED-RESOLUTION DESIGNS OF MINIMUM ABERRATION

Because design size is a practical consideration, the CMR designs presented will be constructed from smallest 2^{K-P} MR designs, augmented with the necessary star and center points. Although smaller designs for estimating the CMR model exist, they, like the smallest composite designs of Hartley (1959), are asymmetrical and inefficient. The CMR designs that we propose are symmetrical within the experimental region.

Satisfaction of the three MR design criteria does not guarantee uniqueness of a 2^{K-P} design. Accordingly, additional criteria will be imposed to determine which MR design is "best." Our second criterion (which will be maximized) is the design resolution among the noise factors. As a third and final criterion, we will use design aberration (which will be minimized). We adopt the definition of minimum aberration stated by Fries and Hunter (1980). That is, if, among all designs of maximum resolution, a design minimizes the number of words of minimum length in the defining relation, then it is a *minimum aberration design.* If the number of words in the defining relation of minimum length are equal for several designs and aberration is the criterion, the search continues by comparing the number of words in the defining relation for the next shortest word length. The process continues until one design is ranked superior to the other. We apply these minimum aberration concepts to find a smallest minimum aberration mixed resolution (MAMR) design over the set of all 2^{K-P} designs. See Franklin (1984), Chen and Lin (1991), Chen and Wu (1991), and Chen (1992) for more information on minimum aberration.

For $K \geq 7$ design factors, MAMR designs were found using a FORTRAN-based search procedure (Borkowski 1992a). The algorithm begins by sequentially generating all 2^{K-2} designs (up through permutation of factors). The wordlength pattern associated with each design's defining relation is compared to the currently best wordlength pattern, and on improvement, the current design becomes the new "best" design. Once the set of 2^{K-2} designs is exhausted, an MAMR design of size 2^{K-2} has been found. The procedure continues with the next largest fraction, and the algorithm either finds (a) an MAMR design of size 2^{K-3} or (b) that no MR design exists of size 2^{K-3}. If (b), then the procedure terminates and the MAMR of 2^{K-2} is chosen. If (a), then the procedure continues for $P = 4, 5, \ldots$ until the smallest MAMR design of size 2^{K-P} is found. The CMR design is then formed by adding center points and star points corresponding to the signal factors.

In Table 1, the MAMR designs are labeled according to the total number of design factors $K = 4, 5, \ldots, 12$ and a reference letter $A, B, \ldots$. The generators for these designs are found in Table 1. Single letters in the "Signal factors" and "Noise factors" columns correspond to the $K-P$ columns generating the full factorial design in $K-P$ factors. The levels of the remaining P factors are generated from the P defining interactions of the 2^{K-P} fractional factorial design. Note that an MR design could be a minimum aberration design for several cases. For example, Design 7A defines the minimum aberration 2^{7-2} design for both $C = 2$ and $C = 3$ signal factors given $K = 7$ design factors. Although this design can accommodate at most three signal factors, the experimenter has the option of treating any one of the three signal factors as a noise factor. Doing so yields a minimum aberration design for two signal and five noise factors. This will be true for any designs with the same design label.

An example of the construction of a CMR design using Table 1 will now be presented. Suppose an experimenter requires a CMR design with four signal factors (A, B, C, D) and three noise factors (E, F, G). Because $C = 4$ and $U = 3$, CMR Design 7B was selected from Table 1. The factorial portion of the CMR design is based on the 2^{7-2} fractional factorial design with the given alias structure. That is, a 2^5 design is set up in five factors A, B, C, D, and E. The remaining two factor levels are formed by taking the products F = ABCD and G = ABCDE. The signal factors A, B, C, and D are aliased with four-way or higher interaction effects, and the noise factors E, F, and G are aliased with two-way or higher interaction effects. Therefore, the design is Resolution V among the four signal factors and Resolution III among the three noise factors. Eight star points and four center points are added to complete the CMR design. This 44-point design is given in Figure 2 with the column labels x_1, x_2, x_3, x_4 and z_1, z_2, z_3 corresponding to the sig-

Table 1. Generators of Mixed-Resolution Designs

Design #	Factors K	Factors C	Fraction	Resolution	Signal factors	Noise factors			
4A	4	2	2^4	—	A B	C	D		
5A	5	2-3	2^{5-1}	V	A B C	D	E = ABCD		
6A	6	2-4	2^{6-1}	VI	A B C D	E	F = ABCDE		
7A	7	2-3	2^{7-2}	IV	A B C	D	E	F = ABCE	G = ABCD
7B	7	4	2^{7-2}	III	A B C D	E	F = ABCD	G = ABCDE	
7C	7	5	2^{7-1}	VII	A B C D E	F	G = ABCDEF		
8A	8	2	2^{8-3}	IV	A B	C	D	E	F = ABCE
						G = ABCD	H = ABDE		
8B	8	3	2^{8-3}	III	A B C	D	E	F = ABCE	G = ABCD
						H = ABCDE			
8C	8	4-6	2^{8-2}	V	A B C D E F	G = CDEF	H = ABEF		
9A	9	2	2^{9-4}	IV	A B	C	D	E	F = CDE
						G = ABCE	H = ABDE	J = ABCD	
9B	9	3	2^{9-4}	III	A B C	D	E	F = DE	G = ABCD
						H = ABCE	J = ABCDE		
9C	9	4-5	2^{9-3}	IV	A B C D E	F	G = ACDEF	H = BDEF	J = ABCF
9D	9	6	2^{9-3}	III	A B C D E F	G = CDEF	H = ABEF	J = ABCD	
9E	9	7	2^{9-2}	VI	A B C D E F G	H = CDEFG	J = ABEFG		
10A	10	2	2^{10-4}	IV	A B	C	D	E	F
						G = BCEF	H = BDEF	J = ACDF	K = ACDE
10B	10	3-4	2^{10-4}	IV	A B C D	E	F	G = ABDE	H = ABDF
						J = BCEF	K = ACDEF		
10C	10	5	2^{10-4}	III	A B C D E	F	G = CDEF	H = ABCEF	J = ABDF
						K = ABCE			
10D	10	6-8	2^{10-3}	V	A B C D E F G	J = ABEFG	K = ABCDEFG		
					H = CDEFG				
11A	11	2-3	2^{11-5}	IV	A B C	D	E	F	G = ABCF
						H = BDEF	J = ABCD	K = ABCE	L = ACDEF
11B	11	4	2^{11-5}	III	A B C D	E	F	G = CDEF	H = ABEF
						J = ABCDE	K = ABCDF	L = ABCD	
11C	11	5-9	2^{11-4}	V	A B C D E F G				
					H = DEFG	K = ACEG	L = ABDF		
					J = BCFG				
12A	12	2	2^{12-6}	IV	A B	C	D	E	F
						G = ABCD	H = ABDF	J = ABCF	K = ABDE
						L = ABCE	M = ABEF		
12B	12	3	2^{12-6}	III	A B C	D	E	F	G = ABCD
						H = ABCE	J = DEF	K = ABCDEF	L = ABCDE
						M = ABCF			
12C	12	4-8	2^{12-5}	IV	A B C D E F G	J = BCFG	K = ACEG	L = ABDG	M = DEFG
					H = ABCDEFG				
12D	12	9	2^{12-5}	III	A B C D E F G				
					H = ACEG	K = BCDE	L = DEFG	M = BCFG	
					J = ABDF				
12E	12	10	2^{12-4}	VI	A B C D E F G H				
					J = DEFGH	L = ACEGH	M = ABDFH		
					K = BCFGH				

NOTE: *K* is the total number of signal and noise design variables. *C* is the number of signal design variables.

nal factors A, B, C, and D and the noise factors E, F, and G, respectively.

4. DESIGN SIZE COMPARISONS

Using the two "Total points" columns in Table 2, the sizes of our proposed CMR designs (given C signal and U noise variables) can be compared to the sizes of the corresponding Taguchi crossed arrays. CMR designs that maximize G efficiency (see Sec. 5) require no center points except where a $\diamond$ indicates that one center point is needed to maximize G efficiency. The comparison symbols C, c, T and t in the "Size" column indicate, respectively, whether the CMR (C or c) or the Taguchi design (T or t) is the smaller design. Uppercase (C or T) indicates that the size difference is greater than 10%, and lowercase (c or t) indicates that the size difference is less than 10%. Table 2 shows that many CMR designs are superior to or competitive with the corresponding Taguchi designs with respect to design size. Only when there are many signal factors will the Taguchi designs require fewer points. This is not surprising because, to fit the CMR model in (2), a design must contain additional points that can provide parameter estimates of the signal variable interaction effects that are excluded from the Taguchi crossed-array model in (1).

The entries in the "Reduction" column indicate whether the fraction is $\frac{1}{4}$, $\frac{1}{2}$, or equal in size to the smallest fraction required by a classical central composite design whose factorial part is a design of at least Resolution V. For many

	Signal				Noise		
	x_1	x_2	x_3	x_4	z_1	z_2	z_3
1	-1	-1	-1	-1	-1	1	-1
2	-1	-1	-1	-1	1	1	1
3	-1	-1	-1	1	-1	-1	1
4	-1	-1	-1	1	1	-1	-1
5	-1	-1	1	-1	-1	-1	1
6	-1	-1	1	-1	1	-1	-1
7	-1	-1	1	1	-1	1	-1
8	-1	-1	1	1	1	1	1
9	-1	1	-1	-1	-1	-1	1
10	-1	1	-1	-1	1	-1	-1
11	-1	1	-1	1	-1	1	-1
12	-1	1	-1	1	1	1	1
13	-1	1	1	-1	-1	1	-1
14	-1	1	1	-1	1	1	1
15	-1	1	1	1	-1	-1	1
16	-1	1	1	1	1	-1	-1
17	1	-1	-1	-1	-1	-1	1
18	1	-1	-1	-1	1	-1	-1
19	1	-1	-1	1	-1	1	-1
20	1	-1	-1	1	1	1	1
21	1	-1	1	-1	-1	1	-1
22	1	-1	1	-1	1	1	1
23	1	-1	1	1	-1	-1	1
24	1	-1	1	1	1	-1	-1
25	1	1	-1	-1	-1	1	-1
26	1	1	-1	-1	1	1	1
27	1	1	-1	1	-1	-1	1
28	1	1	-1	1	1	-1	-1
29	1	1	1	-1	-1	-1	1
30	1	1	1	-1	1	-1	-1
31	1	1	1	1	-1	1	-1
32	1	1	1	1	1	1	1
33	-1	0	0	0	0	0	0
34	1	0	0	0	0	0	0
35	0	-1	0	0	0	0	0
36	0	1	0	0	0	0	0
37	0	0	-1	0	0	0	0
38	0	0	1	0	0	0	0
39	0	0	0	-1	0	0	0
40	0	0	0	1	0	0	0
41	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0

Figure 2. A 44-Point CMR Design With Four Signal and Three Noise Factors.

combinations of C and U, use of a CMR design could lead to a reduction of 32 to 128 experimental runs. This supports the claim that CMR designs can be more economical in size than central composite designs formed from fractional factorial designs of at least Resolution V.

5. COMPOSITE MIXED-RESOLUTION DESIGN EFFICIENCIES

In addition to looking at design size or at limitations due to time, money, or physical constraints, design optimality criteria are often used in design evaluation. If several alternative designs are proposed, then their design efficiencies can be compared to aid in the choice of design. Two of the most commonly used criteria are based on

$$D = |X'X| \quad \text{and} \quad G = \max_{x \in X} N f'(x)(X'X)^{-1} f(x),$$

where $f(x) = [f_1(x), \ldots, f_k(x)]$ is a vector of k real-valued functions based on the model terms and X is the design space. For example,

$$f(x) = [1, x_1, \ldots, x_p, x_1^2, \ldots, x_p^2, x_1x_2, \ldots, x_{p-1}x_p]$$

for a p-variable quadratic model. The D efficiency is $(D/D_{\text{optimal}})^{1/k}$ and the G efficiency is k/G, where k is the number of model parameters. Thus, to calculate the D efficiency, the D_{optimal} value must be obtained. This is discussed in the Appendix.

Efficiencies for CMR designs were determined assuming the CMR model in (2) on a hypercube design region. The values are summarized in Table 2 for $K \leq 12$ factors. The methods used to calculate the D and G efficiencies are summarized in the Appendix. We do not show the efficiencies for Taguchi designs because they are all factorial arrays. Factorial arrays are 100% efficient for estimating a saturated model and some reduced models. For example, the crossed array in Figure 1 is 100% efficient for estimating Model (1). It is 0% efficient for estimating Model (2), however, because it can estimate no interactions between signal factors. Efficiency measures are highly model dependent as well as region dependent.

CMR designs performed consistently well with respect to D efficiency. Two reasons help to explain this. First, there are, in general, relatively few quadratic term parameters in (2). Second, a design consisting of the F factorial points is known to be D optimal with respect to the remaining parameters.

The G efficiencies, however, were much more variable than the D efficiencies. We see that G efficiencies are inversely related to $F/2C$, the ratio of the number of factorial points to the number of star points. If the ratio $F/2C$ is large, then the G efficiency is small. Note that if $F = 2^f$ for a CMR design with i signal factors and $F = 2^{f+1}$ for a CMR design with $(i+1)$ signal factors, then a significant decrease in G efficiency occurs. For example, a larger CMR design is required as C changes from $C = 3$ to $C = 4$ when $K = 8$ or 9. The G efficiency drops accordingly: (83.8% $\rightarrow$ 61.5%) for $K = 8$ and (86.4% $\rightarrow$ 66.3%) for $K = 9$. This decrease in G efficiency can be prevented by decreasing the $F/2C$ ratio through star-point replication. Although replicating star points would require $2C, 4C, \ldots$ additional star points, there are many cases when the G efficiencies dramatically increase (especially when F is large relative to N). For example, when $K = 9$ and $C = 4$, if we add one star-point replicate, then the G efficiency increases from 66.3% to 82.7%. For standard central composite designs, Borkowski (1995a,b) showed how star-point replication affects the prediction variance and how star-point replication improves design efficiency.

For CMR designs with relatively few signal factors, the G efficiency is maximized with one or two sets of star points, and then it decreases as the number of star-point replicates increases. When there are many signal factors, however, the G efficiency increases with star-point replication until an optimal number of replications is achieved, and then it decreases. Table 2 includes the optimal star-point replication number and the corresponding G efficiency (with $\diamondsuit$ indicating one center point, otherwise zero center points). The † and ‡ entries in the "SP reps" column of Table 2

Table 2. A Design Size Comparison and CMR Design Efficiencies

Number of factors			Taguchi design		Size	CMR design			Design efficiencies			
Total K	*Signal C*	*Noise U*	*Inner × outer*	*Total points*	*Smaller design*	*Total points*	*Reduction*	*Parameters*	*D eff*	*G eff*	*SP reps*	*Max G eff*
4	2	2	9 × 4	36	C	21◇	=	12	91.5	83.5	1◇	83.5
5	2	3	9 × 4	36	C	21◇	=	15	90.2	88.3	1◇	88.3
5	3	2	9 × 4	36	C	22	=	18	88.9	87.1	1	87.1
6	2	4	9 × 8	72	C	37◇	=	18	93.0	74.1	2◇	89.2
6	3	3	9 × 4	36	t	38	=	22	93.4	76.8	2	85.7
6	4	2	9 × 4	36	t	40	=	25	91.6	74.7	2	81.1
7	2	5	9 × 8	72	C	37◇	$\frac{1}{2}$	21	93.2	79.0	2◇	87.4
7	3	4	9 × 8	72	C	38	$\frac{1}{2}$	26	93.3	80.7	2†	83.4
7	4	3	9 × 4	36	t	40	$\frac{1}{2}$	30	91.1	78.1	2†	78.3
7	5	2	18 × 4	72	t	74	=	33	89.8	52.7	3†	77.8
8	2	6	9 × 8	72	C	37◇	$\frac{1}{2}$	24	93.2	83.0	2†	86.1
8	3	5	9 × 8	72	C	38	$\frac{1}{2}$	30	93.0	83.8	1	83.8
8	4	4	9 × 8	72	=	72	=	35	92.6	61.5	3†	83.0
8	5	3	18 × 4	72	t	74	=	39	90.6	58.0	3†	78.4
8	6	2	18 × 4	72	t	76	=	42	88.4	55.3	3†	74.3
9	2	7	9 × 8	72	C	37◇	$\frac{1}{4}$	27	93.2	86.5	2†	87.2
9	3	6	9 × 8	72	C	38	$\frac{1}{4}$	34	92.8	86.4	1	86.4
9	4	5	9 × 8	72	=	72	$\frac{1}{2}$	40	93.1	66.3	2	82.7
9	5	4	18 × 8	144	C	74	$\frac{1}{2}$	45	91.1	62.6	2	79.1
9	6	3	18 × 4	72	t	76	$\frac{1}{2}$	49	88.9	60.0	2	76.0
9	7	2	18 × 4	72	T	142	=	52	86.7	35.5	5†	72.6
10	2	8	9 × 12	108	C	69◇	$\frac{1}{2}$	30	95.0	67.8	3◇†	89.6
10	3	7	9 × 8	71◇	c	70	$\frac{1}{2}$	38	94.9	72.1◇	2	88.6
10	4	6	9 × 8	72	=	72	$\frac{1}{2}$	45	93.4	70.7	2	84.3
10	5	5	18 × 8	144	C	74	$\frac{1}{2}$	51	91.3	66.7	2	80.4
10	6	4	18 × 8	144	c	140	=	56	89.8	42.0	4†	77.2
10	7	3	18 × 4	72	T	142	=	60	88.1	39.4	4†	74.2
10	8	2	27 × 4	108	T	144	=	63	86.4	37.2	4†	70.8
11	2	9	9 × 12	108	C	69◇	$\frac{1}{2}$	33	95.3	71.1	2	89.6
11	3	8	9 × 12	108	C	70	$\frac{1}{2}$	42	95.1	75.0	2	90.4
11	4	7	9 × 8	72	=	72	$\frac{1}{2}$	50	93.5	74.6	2	85.6
11	5	6	18 × 8	144	c	138	=	57	92.2	48.0	4†	81.4
11	6	5	18 × 8	144	c	140	=	63	90.7	45.7	4†	77.9
11	7	4	18 × 8	144	c	142	=	68	89.1	43.1	4†	74.9
11	8	3	27 × 4	108	T	144	=	72	87.5	41.1	4†	72.3
11	9	2	27 × 4	108	T	146	=	75	86.0	38.4	4†	69.4
12	2	10	9 × 12	108	C	69◇	$\frac{1}{4}$	36	95.5	74.2	2◇	92.0
12	3	9	9 × 12	108	C	70	$\frac{1}{4}$	46	95.3	77.6	2	90.5
12	4	8	9 × 12	108	T	137◇	$\frac{1}{2}$	55	94.2	53.0	4†	86.7
12	5	7	18 × 8	144	c	138	$\frac{1}{2}$	63	92.8	51.2	4†	82.3
12	6	6	18 × 8	144	c	140	$\frac{1}{2}$	70	91.4	49.2	3	78.8
12	7	5	18 × 8	144	c	142	$\frac{1}{2}$	76	89.9	46.5	4†	75.4
12	8	4	27 × 8	216	C	144	$\frac{1}{2}$	81	88.4	44.8	4†	72.6
12	9	3	27 × 4	108	T	146	$\frac{1}{2}$	85	86.9	42.1	4†	70.2
12	10	2	27 × 4	108	T	276	=	88	84.6	23.2	7‡	68.1

NOTE: C, c, T, t indicate whether the CMR (C or c) or the Taguchi design (T or t) is the smaller design. Uppercase (C or T) indicates a size difference > 10%. Lowercase (c or t) indicates a size difference < 10%. ◇ indicates that one center point is needed for maximum *G* efficiency. "SP reps" is the number of star-point replications maximizing *G* efficiency. † and ‡ indicate that there is <5% efficiency gain from the design with one (if †) or two (if ‡) fewer replicates than the value in the "SP reps" column.

indicate that designs with one (if †) or two (if ‡) fewer star-point replicates than the optimum will have G efficiencies within 5% of the maximum G efficiency. For example, when $K = 8$ and $C = 4$, three star-point replicates maximize the G efficiency. The † indicates, however, that the G efficiency with two replicates ($G_{\text{eff}} = 80.7\%$) is within 5% of the maximum (83.0%). In such cases, the benefit of a small increase in efficiency may not outweigh the cost of the $2C$ or $4C$ additional points.

For those design cases in Table 2 when four or more star-point replicates maximize the G efficiency, the first two or three replicates account for most of the increase in the G efficiency. We suggest that, after two star-point replicates, one should consider using a computer algorithm to increase efficiency through design augmentation.

6. CONCLUSIONS

In this article, experimental designs that can be used for achieving a robust process are discussed. We propose the use of CMR designs as alternatives to the Taguchi robustness designs. The differences between the single-factor array structure of CMR designs and the crossed-array structure of Taguchi designs were discussed, and their corresponding response surface models were compared within a response surface methodology framework. We indicated potential advantages of CMR designs over the competing Taguchi designs with respect to achieving a robust process. The primary advantage is a CMR design's structure because it allows the estimation of potentially important interaction effects among the signal process factors, which, in turn, can be used to determine optimal or near-optimal process-variable levels.

Extensions of the results in this article to $K = 17$ design factors and determination of efficiencies through development of optimal CMR designs was shown by Borkowski (1992b; 1995c). For additional guidance regarding implementation of these designs, we suggest referring to Lucas (1989, 1994).

APPENDIX: CALCULATING *G* AND *D* EFFICIENCIES

To calculate D efficiencies, the D_{optimal} value must be obtained from an optimum design. We now describe the three-step procedure used to find an optimum design assuming Model (2) in a hypercube design region. We (1) conjecture sets of points supporting an optimum design, (2) find optimum weights for each of these sets, and (3) check whether the design is G optimal by verifying that the maximum value for G is the number of model parameters. If it is G optimal, then it is also D optimal by the Kiefer–Wolfowitz (1960) equivalence theorem, and hence the D criterion for this design is D_{optimal}.

For step (1), we consider specific sets of *barycentric* points that can support an optimum CMR design. A barycenter of depth j on q coordinates is a point with j coordinates equal to 0 and $q-j$ coordinates equal to ± 1. The set of barycenters of depth j on q coordinates is denoted $J_q(j)$, and the union of the sets of barycenters $J_q = \cup_{j=0}^{q} J_q(j) =$ the 3^q factorial array of q-tuples with coordinates 0 or ± 1. J_q is the *complete barycentric set* with q coordinates.

Farrell, Kiefer, and Walbran (1967) showed that an optimum design for a quadratic model in a hypercube design region can be found by application of appropriate nonnegative weights to the sets $J_q(0), J_q(1)$, and $J_q(q)$. That is, $J_q(0) \cup J_q(1) \cup J_q(q)$ *support* an optimum design. We conjectured an analogous result for an optimum design for the CMR model in (2). For $1 \le a \le C$ and for $1 \le b \le U$, we define the set of barycenters of depth (a, b) as the product of $J_C(a) \times J_U(b)$. The set of barycenters of depth (a, b) will be denoted as $J_{C+U}(a, b)$ and the union of all such sets $\cup_{a=0}^{C} \cup_{b=0}^{U} J_{C+U}(a, b) = J_{C+U}$. For the CMR model with $K = C + U$ variables, we conjectured in step (1) that the union of $J_K(0,0), J_K(1,0)$, and $J_K(C,0)$ supports an optimum design (Borkowski 1992b, 1995c).

In step (2), weights $\{\alpha_1, \alpha_2, \alpha_3\}$, which maximize the generalized variance criterion $D = |M|$ (where M is the moment matrix), are assigned to $J_K(0,0), J_K(1,0)$, and $J_K(C,0)$, respectively. These weights were found by numerically solving a system of partial derivatives of the closed-form expression for $D = |M|$.

In step (3) we showed that for each choice of C and U, the set of weights $\{\alpha_1, \alpha_2, \alpha_3\}$ yielded a G-optimal design. This is accomplished by evaluating the relative prediction variance G over the set of barycenters and taking the maximum value. Borkowski (1995c) showed that the relative prediction variance function need only be evaluated over a set of $C + 1$ barycentric points to achieve the global maximum. For all designs with $K \le 12$ factors, $G = k =$ the number of model parameters. Thus, from the equivalence theorem of Kiefer and Wolfowitz (1960), we have found a D-optimum and a G-optimum design.

See Borkowski (1995c) for D_{optimal} values and design weights for optimum CMR designs. These D_{optimal} values are essential in the calculation of D efficiencies for CMR designs. Calculation of D efficiencies is made easy by applying the closed-form expression of $|X'X|$ for CMR designs (Borkowski 1992b, 1995c), which we show as Theorem 1:

Theorem 1. For a CMR design with C signal and U noise factors and the design region being the hypercube, the generalized prediction variance function is

$$|X'X| = (F+2)^C \cdot F^{\binom{C}{2}+U(C+1)} \cdot 2^{C-1} \times \{(2+CF)(F+2C+N_0) - C(F+2)^2\}, \quad \text{(A.1)}$$

where F is the number of MR design points and N_0 is the number of center points. D efficiencies for CMR designs are contained in Table 2.

When evaluating G efficiencies, the relative prediction variance function must be maximized over the experimental region. A closed-form expression for the relative prediction variance of CMR designs (Borkowski 1992b) also exists and is given by Theorem 2:

Theorem 2. For a CMR design with C signal and U noise factors and the design region being the hypercube, the relative prediction variance function is

$$\begin{aligned} &f(x)'(X'X)^{-1}f(x) \\ &= \alpha_{11} + \left(2\alpha_{12} + \frac{1}{F+2}\right)\sum_{i=1}^{C} x_i^2 \\ &+ \frac{1}{2}\left(\sum_{i=1}^{C} x_i^4 - \alpha_{22}\left(\sum_{i=1}^{C} x_i^2\right)^2\right) \\ &+ \frac{1}{F}\sum_{i=1}^{C-1}\sum_{j=i+1}^{C} x_i^2 x_j^2 + \frac{1}{F}\sum_{k=1}^{U} z_k^2 \\ &+ \frac{1}{F}\sum_{i=1}^{C}\sum_{k=1}^{U} x_i^2 z_k^2, \end{aligned} \tag{A.2}$$

where F is the number of MR design points and

$$\begin{aligned} \alpha_{11} &= \frac{CF+2}{2N+CNF-C(F+2)^2} \\ \alpha_{12} &= -\frac{F+2}{2N+CNF-C(F+2)^2} \\ \alpha_{22} &= \frac{NF-(F+2)^2}{2N+CNF-C(F+2)^2}. \end{aligned}$$

Borkowski (1992b) showed that the maximum prediction variance can be found by maximizing (4) over the set of barycenters. G efficiencies are contained in Table 2.

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REFERENCES

Borkowski, J. J. (1992a), "Computer Algorithms for Generating Two-Level Mixed Resolution Designs of Minimum Aberration," Technical Report 10-10-92, Montana State University, Dept. of Mathematical Sciences.

——— (1992b), "The Evaluation of Mixed Resolution Designs," unpublished Ph.D thesis, University of Delaware, Dept of Mathematical Sciences.

——— (1995a), "Spherical Prediction Variance Properties of Central Composite and Box–Behnken Designs," *Technometrics*, 37, 399–410.

——— (1995b), "Finding Maximum G-criterion Values for Central Composite Designs on the Hypercube," *Communications in Statistics—Theory and Methods*, 24, 2041–2058.

——— (1995c), "Optimal Linear-Quadratic (LQ) Model-Based Designs," unpublished manuscript.

Box, G. E. P., and Fung, C. A. (1986), "Studies in Quality Improvement: Minimizing Transmitted Variation by Parameter Designs," Report 8, University of Wisconsin-Madison, Center for Quality and Productivity Improvement.

Box, G. E. P., and Jones, S. (1990), "Designing Products That Are Robust To Environment," Report 56, University of Wisconsin-Madison, Center for Quality and Productivity Improvement.

Box, G. E. P., and Wilson, K. B. (1951), "On the Experimental Attainment of Optimum Conditions," *Journal of the Royal Statistical Society*, Ser. B, 13, 1–45.

Chen, J. (1992), "Some Results on 2^{n-k} Fractional Factorial Designs and Search for Minimum Aberration Designs," *The Annals of Statistics*, 20, 2124–2141.

Chen, J., and Lin, D. K. J. (1991), "On the Identity Relationships of 2^{k-p} Designs," *Journal of Statistical Planning and Inference*, 28, 95–98.

Chen, J., and Wu, C. F. J. (1991), "Some Results on s^{n-k} Fractional Factorial Designs With Minimum Aberration or Optimal Moments," *The Annals of Statistics*, 19, 1028–1041.

Cochran, W. G., and Cox, G. M. (1957), "Experimental Design," New York: Wiley.

Deming, W. E. (1946), *The Statistical Adjustment of Data,* New York: Wiley.

Farrell, R. H., Kiefer, J., and Walbran, A. (1967), "Optimum Multivariate Designs," in *The Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, Berkeley: University of California Press, pp. 113–138.

Franklin, M. F. (1984), "Constructing Tables of Minimum Aberration p^{n-m} Designs," *Technometrics*, 26, 225–232.

Fries, A., and Hunter, W. G. (1980), "Minimum Aberration 2^{k-p} Designs," *Technometrics*, 22, 601–608.

Hartley, H. O. (1959), "Smallest Composite Designs for Quadratic Response Surfaces," *Biometrics*, 15, 611–624.

Kempthorne, O. (1952), "The Design and Analysis of Experiments," New York: Wiley.

Kiefer, J., and Wolfowitz, J. (1960), "The Equivalence of Two Extremum Problems," *Canadian Journal of Mathematics*, 12, 363–366.

Lucas, J. M. (1989), "Achieving a Robust Process Using Response Surface Methodology," in *Proceedings of the American Statistical Association, Sesquicentennial Invited Paper Sessions*, pp. 579–593.

——— (1994), "How to Achieve a Robust Process Using Response Surface Methodology," *Journal of Quality Technology*, 26, 248–260.

Myers, R. H., Khuri, A. I., and Vining, G. (1992), "Response Surface Alternatives to the Taguchi Robust Parameter Design Approach," *The American Statistician*, 46, 131–139.

Sacks, J., Welch, W. J., Mitchell, T. J., and Wynn, H. P. (1989), "Design and Analysis of Computer Experiments," *Statistical Science*, 4, 409–435.

Shoemaker, A. C., Tsui, K.-L., and Wu, C. F. J. (1991), "Economical Experimentation Methods for Robust Design," *Technometrics*, 33, 415–427.

Taguchi, G. (1991), "System of Experimental Design" (Vols. I and II), Dearborn, MI: American Supplier Institute.

Taguchi, G., and Wu, Y. (1979), "Introduction to Off-line Quality Control," Nagaya, Japan: Central Japan Quality Control Association.

Welch, W. J., Sacks, J., and Schiller, S. B. (1989), "Designs for Computer Experiments," *Technometrics*, 31, 41–47.