PROJECT 2

Math 441

Due: Thursday, Sept. 18

Turn in solutions to all problems except #0. All "by hand" calculations require that you turn in your work with your solutions. All MATLAB calculations must be accompanied by your MATLAB code.

0. If you need practice with MATLAB, check out the lab at the course web page. Do not turn in any work associated with this problem.

1. An airplane flies the (approximately) 2000 mile journey back and forth from Boston to Bozeman. Use the following information to create a linear model to determine the speed of the airplane, s_1, and the constant windspeed, s_2. When flying to Bozeman from Boston, the airplane flies into a constant headwind, and the journey takes about 4 hours and 24 minutes. When returning to Boston and traveling in the same direction as the wind, the flight takes 4 hours and 2 minutes.

   (a) Write your model as $A\boldsymbol{s} = \boldsymbol{d}$.

   (b) How many solutions are there to this system? Justify your answer.

   (c) Solve the model by hand using Gauss elimination. Show your work.

2. Let A be an $m \times n$ matrix and $\boldsymbol{u}$ an $n \times 1$ non-zero vector. Suppose that $A\boldsymbol{u} = 0$. Show that the columns of A are linearly dependent (*Hint:* use the definition of linear independence).

3. Let A be an $m \times n$ matrix and $\boldsymbol{u}$ an $n \times 1$ non-zero vector. Suppose that $A\boldsymbol{u} = 0$. Also suppose that there exists a solution vector $\boldsymbol{x}^*$ such that $A\boldsymbol{x}^* = \boldsymbol{b}$ for an $m \times 1$ vector $\boldsymbol{b}$. Show that there are an infinite number of solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$. What do these solutions look like?

4. **Determinants are multiplicative:** Do problem 1.2.5 in your textbook (*Hint:* Use the fact that $\det(AB) = \det(A)\det(B)$).

5. **Finite difference approximations:** Do problem 1.2.21 (all three parts) in your textbook.

6. **Sum of an arithmetic series:** Do problem 1.3.25 in your textbook.

7. **Plotting in MATLAB:** Use MATLAB to plot the Gaussian $f(x) = e^{-x^2}$ over the range of values $-3 \leq x \leq 3$. In MATLAB, enter:

   - **x = -3:.1:3;**
   - **f = exp(x.^ 2);**
   - **figure(1)**
   - **plot(x,f)**
   - **title('The bell shaped curve, a Gaussian')**

   Include the picture in your solutions.

8. **Taylor series approximation:** In a calculus course, you learn that some functions can be approximated arbitrarily well by a polynomial.

   (a) State the general formula for the Taylor series of an infinitely differentiable function $f(x)$ at $x = 0$.

   (b) Calculate (by hand) $T_3(x)$, the 3rd order Taylor polynomial approximation,

   $$T_3(x) = d + cx + bx^2 + ax^3,$$

   to the Gaussian $f(x) = e^{-x^2}$ at $x = 0$ (that is, find the coefficients d, c, b, a).

   (c) Plot the Taylor polynomial in the same figure that you used in problem #7.

   - **x = -3:.1:3;**
   - **T = d + c*x + b*x.^2 + a*x.^3;**
   - **figure(1)**
   - **hold on** % Allows you to add more plots to the figure
   - **plot(x,T,'r')** % The 'r' makes things red. Type **help plot** for more color codes.
   - **axis([-3 3 -.2 1.4])** % sets the viewing window in the figure
   - **title('The Gaussian and a 3rd order Taylor approximation')**

   Include the picture in your solutions.

   (d) Calculate the relative error of the 3rd order Taylor approximation at the point $x = \frac{1}{2}$,

   $$RelErr(.5) = \frac{|f(.5) - T_3(.5)|}{|f(.5)|}.$$

   Also, calculate the relative error at $x = 1.5$. Why is one so much larger than the other?

9. **Higher order Taylor series approximation:** The exponential function $f(x) = e^x$ has the Taylor series

   $$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{n!}.$$

   For finite n, $f(x)$ is approximated by the Taylor polynomial $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{n!}$.

   (a) Generate a plot in MATLAB of $f(x) = e^x$ and the 4th order Taylor series approximation to it for $-1 \leq x \leq 2$. You do not need to do this by hand. Instead:

   - Download the MATLAB m-file TaylorSeriesForExp.m from the course web page.
   - Make sure that the m-file TaylorSeriesForExp.m is in MATLAB's working directory.
   - In MATLAB, enter **TaylorSeriesForExp**. You can retype this command as often as you'd like to keep seeing the pretty pictures which are plotted in figures 1 and 2.

   (b) After running **TaylorSeriesForExp**, consider the plot of the relative error in figure 2,

   $$RelError_n = \frac{||f(x) - T_n(x)||_2}{||f(x)||_2}$$

   (notice that here I am using the Euclidean norm notation $|| \cdot ||_2$ to define the relative error, and not absolute values as in problem #8 since **x=-3:.1:3**, **f(x)=exp(x)** and $\mathbf{T_n(x)}$ are vectors in MATLAB). Even though the theory says $\lim_{n\to\infty} RelError_n = 0$, what value does the relative error in the figure converge to? Why is this result different than what the theory predicts?

10. **Polynomial Interpolation:** In September of 2004, the journal *Nutritional Epidemiology* published a paper "Coffee Drinking Is Dose-Dependently Related to the Risk of Acute Coronary Events in Middle-Aged Men." One of the conclusions of the paper was that, at least for middle-aged men, there is a quadratic relationship between the amount of coffee consumed and the "rate risk of an acute coronary event." Consider three men from this study and call them individuals $i = 1, 2$ and 3. Let the variable x_i represent the number of cups (1 cup = 8 oz) of coffee consumed each day (on average) by individual i, and let y_i be the rate risk for the same individual. Here's the data:

i	x_i	y_i
1	1	2.7
2	2	2.4
3	4	7.5

(a) Plot the three data points in MATLAB. This will do it:

- **x=[1 2 4]'**
- **y=[2.7 2.4 7.5]'**
- **figure** % creates a new figure window
- **clf** % Clears the figure window
- **plot(x,y,'r*')** % This plots the points using beautiful red asterisks.

(b) Construct the associated Vandemonde matrix A to fit a quadratic to these three data points. You can generate the first column of A using MATLAB's **ones(3,1)** command.

(c) By hand, calculate the determinant of A. Show your work!

(d) Write out the appropriate linear system which must be solved to find the coefficients of the quadratic $y = c + bx + cx^2$.

(e) How many solutions are there to this linear system? Justify your answer.

(f) Solve the linear system (feel free to use MATLAB, and be sure to include the code in your solutions). What are the coefficients of the quadratic?

(g) Plot the quadratic in the same figure window as you plotted the red asterisks. In MATLAB, enter:

- **v=0:.1:5;**
- **f=c + b*v + c*v.^2;**
- **hold on** % This allows you to plot more in the figure window with the red asterisks
- **plot(v,f)** % This plots the quadratic using a beautiful blue curve.

Include the figure in your solutions.