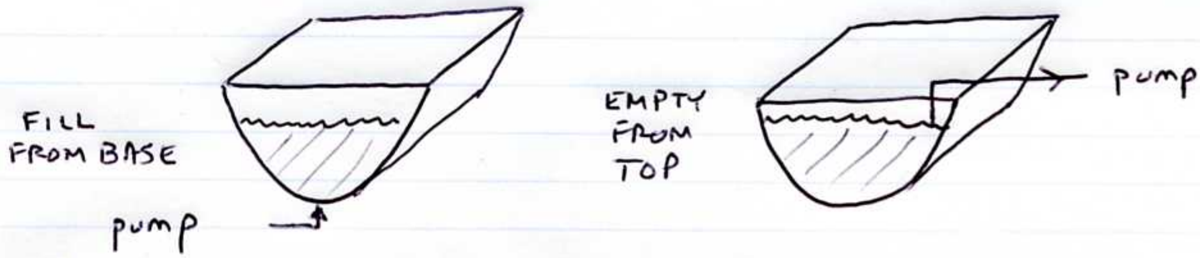
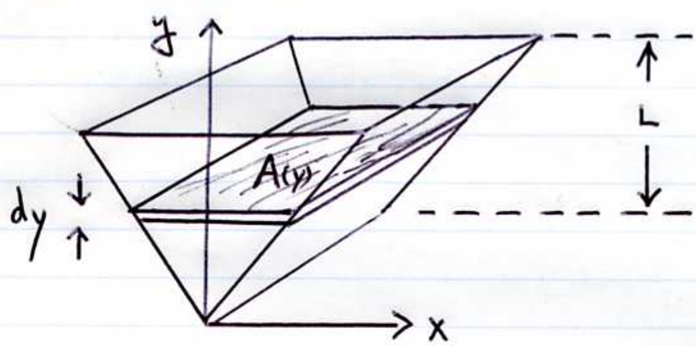
Work Done filling and emptying troughs: Supplementary

Suppose a trough (below) is either to be filled from its base or emptied from the top of its fluid - say water. In either case a pump must do work.



The total work done is obtained by integrating the work done in moving each cross sectional slice of area A



When filling from the base the illustrated slice is moved a distance

$$h = y$$

When pumped from top the slice is moved a distance

$$h = L - y$$

If the density of the fluid is $\rho = 1000\,kg/m^3$ (water) then for filling from the base the amount of work moving the slice is

$$dW = \underbrace{\rho\, A(y)\, dy}_{\text{mass } m}\; g\; h(y) \qquad g = 9.8\,m/sec^2$$

To move all slices when filling from the base

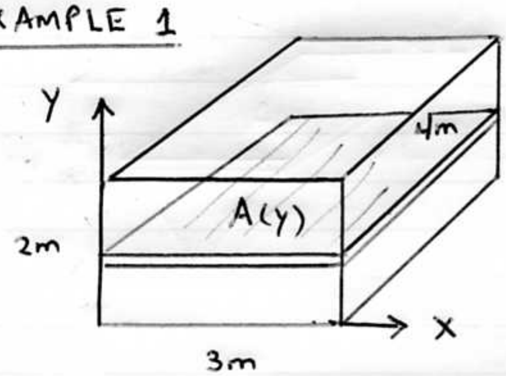
$$W = \int_0^L \rho g\, A(y)\, h(y)\, dy \quad , \quad h(y) = y$$

To move all slices when pumping out from top

$$W = \int_0^L \rho g\, A(y)\, h(y)\, dy \quad , \quad h(y) = L - y.$$

So one needs only find a formula for the cross sectional area $A(y)$ in each case.

EXAMPLE 1



Rectangular trough.

Find the total amount of work done

a) filling from the base
b) pumping from the top

for the trough drawn ($\rho = 1000\ kg/m^3$ water)

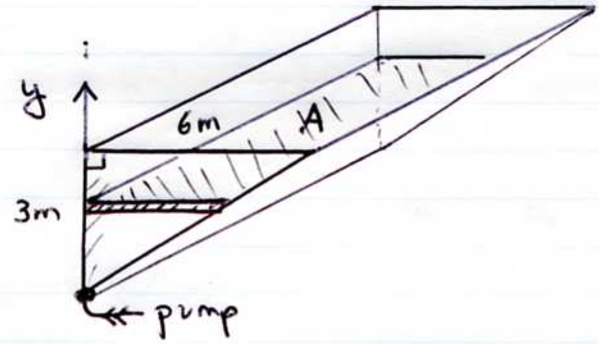
a) $A(y) = 3 \cdot 4 = 12$ $\qquad h(y) = y$ for filling.

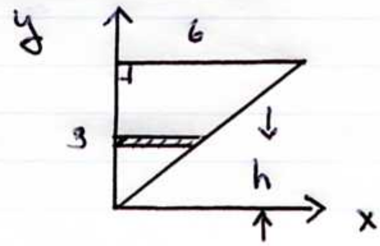
$$W = \int_0^2 \rho g\, A(y)\, h(y)\, dy = \rho g \int_0^2 12y\, dy = \underset{\text{Jole}}{24 \rho g}$$

b) $A(y) = 3 \cdot 4 = 12$ $\qquad h(y) = 2 - y$ for pumping.

$$W = \int_0^2 \rho g\, A(y)\, h(y)\, dy = \rho g \int_0^2 12(2-y)\, dy = 24 \rho g$$

EXAMPLE 2 The trough below is to be filled from its base with water ($\rho = 1000 kg/m^3$). How much work is done to fill the trough?





Solution: Each slice is moved a distance $h(y) = y$.

From the side view the edge has formula $y = \frac{1}{2}x$

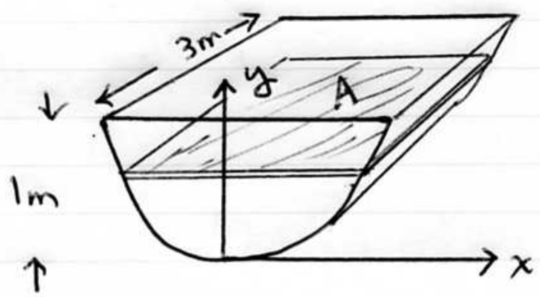
Cross sectional area $A = 6x = 6(2y) = 12y$.

$$W = \int_0^3 \rho g (12y)\, y\, dy = \int_0^3 12\rho g y^2 dy = 108\rho g$$

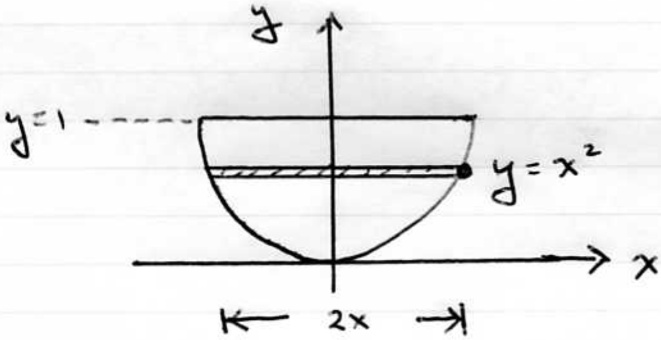
Were we to compute the work done in pumping water out from top, $h(y) = 3-y$, $A(y) = 12y$

$$W = \int_0^3 \rho g (12y)(3-y)\, dy = 54\rho g$$

EXAMPLE 3 The trough below is 1m deep, 3m long and has its edge determined by the parabola $y = x^2$. If it is filled with water ($\rho = 1000\,kg/m^3$) from its base, how much work is done?



Solution: Draw side view of slice:



Area of cross section

$$A = (2x)(3) = 6x.$$

but $y = x^2 \Rightarrow x = \sqrt{y}$ yields

$$A(y) = 6\sqrt{y}$$

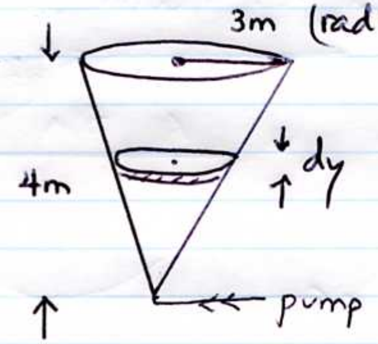
Each slice is moved a distance $h(y) = y$ so

$$W = \int_0^1 \rho g\, A(y)\, h(y)\, dy = \int_0^1 \rho g\, (6\sqrt{y})\, y\, dy = \frac{12}{5}\rho g$$

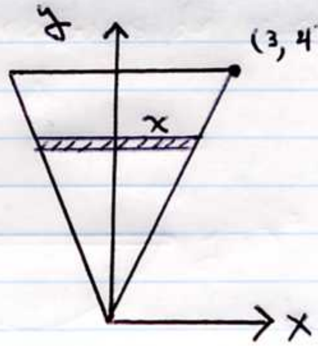
Were one interested in the work done emptying such a trough from the top (pump) $h(y) = 1 - y$

$$W = \int_0^1 \rho g\, 6\sqrt{y}\,(1-y)\, dy = \frac{8}{5}\rho g$$

EXAMPLE 4 The cone below is to be filled with water ($\rho = 1000\,kg/m^3$) from its apex. Determine the work done filling the cone.



Solution



The edge has formula

$$y = \frac{4}{3}x$$

Cross sections are circles of radius x so have area $A = \pi x^2$

But since $y = \frac{4}{3}x$, $x = \frac{3}{4}y$ and

$$A(y) = \pi\left(\frac{3}{4}y\right)^2 = \frac{9}{16}\pi y^2$$

To fill from base, $h(y) = y$ and

$$W = \int_0^4 \rho g \underbrace{\left(\frac{9}{16}\pi y^2\right)}_{A(y)} \underbrace{y}_{h(y)}\, dy = 36\pi\rho g$$