Linear Second Order General Theory:

The following are important Theorems for general second order differential equations:

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

Theorem 1 *Let $p(x)$, $q(x)$ and $f(x)$ be continuous on an interval $I = (a, b)$ containing x_0. Then the solution of the initial value problem*

$$y'' + p(x)y' + q(x)y = f(x) \quad , \quad y(x_0) = y_0 \quad , \quad y'(x_0) = v_0 \tag{2}$$

exists and is unique for all $x \in I$.

Theorem 2 *Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation*

$$a(x)y'' + b(x)y' + c(x)y = 0 \ .$$

If $y_p(x)$ is any particular solution of the nonhomogeneous problem (1) then the general solution of (1) is:

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

where c_1 and c_2 are arbitrary constants.

Definition 1 *Two functions $y_1(x)$ and $y_2(x)$ are linearly independent on an interval $I = (a, b)$ if*

$$c_1y_1(x) + c_2y_2(x) = 0 \quad , \quad for\ all\ x \in I$$

implies $c_1 = c_2 = 0$ for any constants c_1, c_2.

Theorem 3 *Let $y_1(x)$ and $y_2(x)$ be solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \quad , \quad x \in I = (a, b) \tag{3}$$

and define the Wronskian

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

Then y_1 and y_2 are independent if and only if $W(x) \neq 0$ on I.

Note: The last Theorem is more useable than the one in the textbook

Types of Second Order Differential Equations

1) Linear Nonhomogeneous

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

2) Linear Homogeneous

$$a(x)y'' + b(x)y' + c(x)y = 0$$

3) Linear Reducible

$$a(x)y'' + b(x)y' = f(x)$$

4) Linear Constant Coefficient Homogeneous

$$ay'' + by' + cy = 0$$

5) Linear Constant Coefficient Nonhomogeneous

$$ay'' + by' + cy = f(x)$$

6) Euler Equations

$$ax^2y'' + bxy' + cy = 0$$

General Solutions in Constant Coefficient Case:

Substituting $y(x) = e^{rx}$ into

$$ay'' + by' + cy = 0$$

yields the characteristic equation:

$$ar^2 + br + c = 0$$

The general solution depends on the roots r_1, r_2 of this equation:

a) Real distinct roots: $r_1 \neq r_2$

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

b) Real and equal roots: $r_1 = r_2 = r$

$$y(x) = c_1e^{rx} + c_2xe^{rx}$$

c) Complex roots: $r = \alpha \pm i\beta$, $i^2 = -1$

$$y(x) = c_1e^{\alpha x}cos(\beta x) + c_2e^{\alpha x}sin(\beta x)$$

Special Cases for Constant Coefficient Case:

If $b = 0$ then division by a yields

$$y'' + qy = 0$$

for some constant q. Then the solution depends on the sign of q:

a) $q = \omega^2 > 0$

$$y(x) = c_1 sin(\omega x) + c_2 cos(\omega x)$$

b) $q = -\omega^2 > 0$

$$y(x) = c_1 sinh(\omega x) + c_2 cosh(\omega x)$$

where the hyperbolic sine and cosine functions are defined by:

$$sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \quad , \quad cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Complex Solutions in Constant Coefficient Case:

Even if the roots of the characteristic equation are complex (and distinct) the general solution is

$$y(x) = c_1 e^{r_+ x} + c_2 e^{r_- x} \quad , \quad r_\pm = \alpha \pm i\beta$$

In this instance c_1 and c_2 are complex constants. A real solution $y(x)$ is then obtained using DeMoivre's formula:

$$e^{\alpha + i\beta} = e^{\alpha} cos(\beta) + ie^{\alpha} sin(\beta)$$

Second Order Euler Equations:

If a, b, c are constant the Euler equation

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

is solved using the transformation

$$u = ln(x)$$

This transformation of independent variables yields a constant coefficient problem

$$a\frac{d^2y}{du^2} + (b-a)\frac{dy}{du} + cy = 0$$

for $y = Y(u)$. The solution of the orginal problem is $y = Y(ln(x))$.