Chapter 14: Review Questions

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(1) PARTIAL DERIVATIVES AND IMPLICIT DIFFERENTIATION

a) If $f(x,y) = x/(x+y)$ compute $f_x(2,-1)$ and $f_{xy}(2,-1)$.

b) If $f(x,y,z) = x^3y + z^2y$ compute $f_{xz}(1,3,1)$ and $f_{yz}(1,3,1)$.

c) If $f(x,y) = \sqrt{2x+3y}$ compute $f_x(1,1)$ and $f_y(1,1)$.

d) Compute $f_x(1,0)$ if $f(x,y) = e^{x-1}sin(xy)$.

5) Compute $z_x(1,1)$ if $z(x,y)$ is defined implicitly by $x^2z^3 - zy + x - 1 = 0$. Assume $z(x,y) \neq 0$.

(H) e) If $x^2 + z^4 - z - y^2 = 0$, compute z_x and z_{xx} in terms of x, y and z.

(2) LIMITS AND DOMAINS

a) What are the domains of the following functions:

i) $f(x,y) = \sqrt{1-x-y}$ ii) $f(x,y) = log(3 - x^2 - y^2)$ iii) $f(x,y) = (x - \sqrt{1-y^2})^{-1}$

b) Compute the following limits: (Pick any path, i.e. $y=x$ etc)

$$i) \lim_{(x,y)\to(0,0)} \frac{x^3 + xy - yx^2 - y^2}{x + x^2 - y - xy} \quad ii) \lim_{(x,y)\to(0,0)} \frac{x^2 - x + xy - y}{xy - 3x + y^2 - 3y} \quad iii) \lim_{(x,y)\to(0,0)} \frac{sin(xy)}{xy}$$

c) Show the following limits do not exist:

$$i) \lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+3y^2} \quad ii) \lim_{(x,y)\to(1,2)} \frac{x-3+y}{x+1-y} \quad iii) \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

(3) CHAIN RULE

a) Let $f = x^2 + y^2 + 3z$ where $x = 2t + z$ and $y = t - 3z$. Use the chain rule to evaluate f_z and f_t when $(z,t) = (1,1)$.

b) Let $w = x^2 - xy$, $x = u + v$, $y = u - v$. Use the chain rule to evaluate w_u when $(x,y) = (1,0)$.

c) You do not know $f(x,y)$ but do know $f_x = x^2 - y$ and $f_y = x + y$. If $x(t) = t^2 + t$ and $y(t) = 1 - 3t$, what is the rate of change of f in t when $t = 1$?

(4) GRADIENT AND DIRECTIONAL DERIVATIVES

a) Compute $\nabla f(1,2)$ if $f(x,y) = x^2 - \sqrt{y-x}$.

b) What is the directional derivative of $f(x,y) = x^3y - y$ at $(1,1)$ in the direction (1,3). In what direction is f decreasing most rapidly at $(1,1)$?

c) What is the directional derivative of $f(x,y) = xy - y^2$ at (1,1) in a direction toward (4,5)?

d) The temperature in a room is given by $f(x,y) = 100x/(x^2+y^2)$ °F where x, y (feet) are coordinates on the floor. You move along $x = 1$ at $2ft/sec$ in a positive y direction carrying a thermometer. At the moment you pass thru $(x,y) = (1,1)$ at what rate is the tempurature on your thermometer changing?

~~e) Let $f(x,y) = F(r)$ where $r = \sqrt{x^2+y^2}$. Show~~

$$\nabla f = F'(r)\hat{r} \quad , \quad \vec{r} = x\hat{i} + y\hat{j} \quad , \quad \hat{r} = \frac{\vec{r}}{r} \tag{1}$$

(5) GEOMETRY

a) Find two unit vectors perpendicular to the graph of $f(x,y) = x^2 + y^3$ at $(x,y) = (1,1)$.

b) ~~Find two nonparallel vectors tangent to the graph of $f(x,y) = xy + y^2$ at $(x,y) = (1,1)$.~~

c) Find the equation of a plane tangent to the graph of $f(x,y) = x + y + xy^3$ at $(x,y) = (1,1)$.

d) Find the equation of a line, normal to the graph of $f(x,y) = \sqrt{x+3y}$ at $(x,y) = (1,1)$.

e) A line L is normal to the graph of $f(x,y) = 1/(x+2y+1)$ at $(x,y) = (1,1)$. At what point does this line intersect the xy-plane?

f) Find a vector perpendicular to the curve $x^2y = 1$ when $(x,y) = (1,1)$.

g) Find a vector perpendicular to the surface defined by $x^2 - z^3 + zy = 0$ when $(x,y) = (1,0)$.

h) Find a unit vector perpendicular to the $f = 2$ level curve of $f(x,y) = x^2 + y^3$ at $(x,y) = (1,1)$.

i) A curve C is formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the graph $z = f(x,y)$. If $\nabla f(x,y) = (x+y, x-y)$, find a vector tangent to the curve when $(x,y) = (1,0)$ (Hint: $x(t) = cost$).

(6) CRITICAL POINTS AND SECOND DERIVATIVE TEST

a) Find all the critical points of f and classify them

(i)	$f(x,y) = x^2 + 2y^2 - 4x + 4y$	(2,-1) relative minima
(ii)	$f(x,y) = xy - x + y$	(-1,1) saddle
(iii)	$f(x,y) = (2-x)e^{x-y^2}$	(1,0) relative maxima
(iv)	$f(x,y) = x^4 + y^4 - 4xy$	(0,0) saddle
		(1,1),(-1,-1) relative minima
(v)	$f(x,y) = x/y + 8/x - y$	(-4,2) relative maxima
(vi)	$f(x,y) = x^3 + y^3 - 3xy$	(0,0) saddle, (1,1) relative minima
(vii)	$f(x,y) = x^2 - 3xy + 5x - 2y + 6y^2$	(-18/5,-11/15) relative minima

b) If $\nabla f = (f_x, f_y) = (x^2 - y, y - x)$, find and classify all critical points of $f(x,y)$.

(7) MAXIMA AND MINIMA ON A REGION

a) ~~Find the maximum and minimum values of $f(x,y) = 2xy - x - y$ on the square region whose vertices are (0,0), (1,0), (1,1) and (0,1).~~

b) Find the minimum value of $f(x,y) = x^2 - x + y - xy$ on the triangular region whose vertices are (0,0), (2,0) and (2,4).

(8) LAGRANGE MULTIPLIERS

a) Find the maxima and minima of $f(x,y) = x + 2y$ on $x^2 + y^2 = 5$

b) Find the maxima and minima of $f(x,y) = x + 2y$ on $x^2 + y^2 \le 5$ (Look at previous problem).

c) Find the maximum and minimum temperature of $T(x,y) = (x+y)e^{-x^2-y^2}$ on $x^2 + y^2 \le 2$

d) Compute the minimum distance from $xy = 4$ to the origin.

e) Compute the minimum distance from $x + 2y - 3z = 1$ to (0,0,0).

f) The only critical point of $f(x,y) = x^2 + xy + y^2$ is (0,0). Find the maximum of f on the region $x^2 + y^2 \le 1$. (First find max on boundary $x^2+y^2=1$)

g) Find all the extrema of $f(x,y) = x^2 + y^2$ subject to the constraint that $g(x,y) = x^4 + y^4 = 16$.

h) What are the extrema $f(x,y) = x^3 + y^3$ subject to the constraint $g(x,y) = y - x = 0$? Is this point (are these points) a max or min of f for the (x,y) satisfying the constraint?

i) Design a cylindrical can without a top which holds 1 liter of fluid which is made of minimal material (surface area).

(1) PARTIAL DERIVATIVES AND IMPLICIT DIFF.

(a) $f_x(x,y) = \dfrac{y}{(x+y)^2}$ $\qquad f_x(2,-1) = -1$

$f_{xx}(x,y) = -\dfrac{2y}{(x+y)^3}$ $\qquad f_{xx}(2,-1) = 2$

(b) $f_{xz}(x,y,z) = 0$ $\qquad f_{xz}(1,3,1) = 0$

$f_{yz}(x,y,z) = 2z$ $\qquad f_{yz}(1,3,1) = 2$

(c) $f_x = \dfrac{1}{\sqrt{2x+3y}}$ $\qquad f_x(1,1) = \dfrac{1}{\sqrt{5}}$

$f_y = \dfrac{3}{2\sqrt{2x+3y}}$ $\qquad f_y(1,1) = \dfrac{3}{2\sqrt{5}}$

(d) $f_x = e^{(x-1)}(\sin xy + y\cos xy)$, $\quad f_x(1,0) = 0$

(e) Note that $(x,y) = (1,1) \Rightarrow z^3 - z = 0$ so that $z(1,1) = 0, \pm 1$. Implicit differentiation yields

$$z_x(1,1) = \left.\frac{2xz^3 + 1}{y - 3x^2z^2}\right|_{(x,y,z)=(1,1,\pm 1)} = \frac{1}{2}, -\frac{3}{2}$$

(f) Implicit Diff in x

$$2x + 4z^3 z_x - z_x = 0 \qquad z_x = \frac{2x}{(1-4z^3)}$$

Differentiation eqn above in x again.

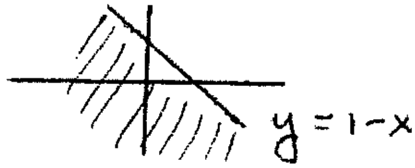
$$2 + 12z^2 z_x^2 + (4z^3-1)z_{xx} = 0$$

and solve for z_{xx}

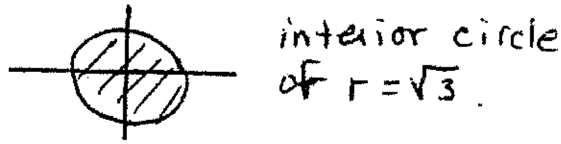
$$z_{xx} = \frac{2 + 12z^2 z_x^2}{(1-4z^3)} = -2\,\frac{(16z^6 - 8z^3 + 1 + 24x^2z^2)}{(4z^3-1)^3}$$

(2) LIMITS AND DOMAINS

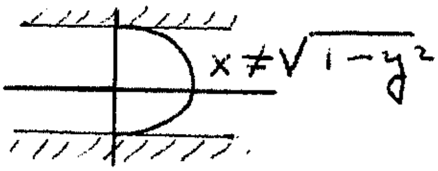
(a) (i) $D = \{(x,y) : 1 - x - y \geq 0\}$



(ii) $D = \{(x,y) : x^2 + y^2 < 3\}$



(iii) $D = \{(x,y) : y^2 \leq 1,\ x \neq \sqrt{1-y^2}\}$



(b) Since the limits exist, can evaluate on any path.

(i) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \dfrac{x^3}{x+x^2} = 0$

(ii) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \dfrac{x^2-x}{-3x} = \dfrac{1}{3}$

(iii) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \dfrac{\sin(x^2)}{x^2} = 1$

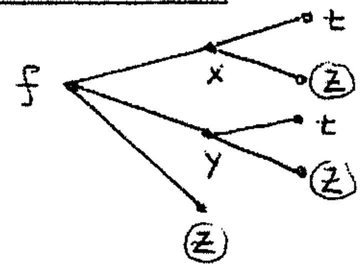
c) Need to show limits on 2 paths have diff. values.

(i) $\lim_{x\to 0} f(x, kx) = \dfrac{k}{2+3k^2}$ diff for diff k vals.

(ii) $\left.\begin{array}{l} \lim_{y\to 2} f(1,y) = \lim_{y\to 2} \dfrac{(-2+y)}{(2-y)} = -1 \\ \lim_{x\to 1} f(x,2) = \lim_{x\to 1} \dfrac{x-1}{x-1} = +1 \end{array}\right\}$ different!

(iii) $\lim_{x\to 0} f(x, kx^2) = \dfrac{k}{1+k^2}$ diffemt for diff $y = kx^2$ paths.

(3) CHAIN RULE

(a) 

$F(z,t) = f(x,y,z)$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$$

$$\frac{\partial F}{\partial z} = 2x \cdot 1 + 2y\cdot(-3) + 3$$

and at $(z,t) = (1,1)$ we have $x = 3$ $y = -2$ so $F_z(1,1) = 21$. Likewise $F_t = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}\Big|_{(1,1)} = 8$.

(b) $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = (2x-y)\cdot 1 - x(1)$$

evaluate at $(x,y) = (1,0)$ yields $w_u =$.

(c)
$$\frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} = (x^2-y)(2t+1) + (x+y)(-3)$$

At $t=1$ one has $x = 2$, $y = -2$ thus we evaluate the above expression at $(t,x,y) = (1,2,-2)$:

$$\frac{df}{dt}\Big|_{t=1} = (2^2+2)(3) + (\cancel{2-2})(-3) = 18$$

(4) GRADIENT AND DIRECTIONAL DERIVATIVES

(a) $\vec{\nabla} f = \left(2x + \frac{1}{2\sqrt{y-x}}, -\frac{1}{2\sqrt{y-x}}\right)$ $\qquad \vec{\nabla} f(2,1) = \langle \frac{5}{2}, -\frac{1}{2}\rangle$

(b) First $\vec{\nabla} f = \langle 3x^2y, x^3-1\rangle$, $\vec{\nabla} f(1,1) = \langle 3, 0\rangle$.
Direction $\vec{u} = \langle 1, 3\rangle$, unit direction $\hat{u} = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$

$$D_{\hat{u}} f(1,1) = \vec{\nabla} f(1,1)\cdot\hat{u} = \frac{3}{\sqrt{10}}$$

Direction f decreasing most rapidly is $-\vec{\nabla} f(1,1) = \langle -3, 0\rangle$.

c) Directional derivative from (1,1) "toward" (4,5).

$\vec{\nabla} f = \langle y, x - 2y \rangle \qquad \vec{\nabla} f(1,1) = \langle 1, -1 \rangle$

direction vector $Q(4,5) - P(1,1) = \langle 3,4 \rangle = \vec{u} \Rightarrow \hat{u} = \frac{1}{5}\langle 3,4 \rangle$ is unit direction vector.

$$D_{\hat{u}} f(1,1) = \vec{\nabla} f(1,1) \cdot \hat{u} = \frac{1}{5}(3-4) = -\frac{1}{5}$$

d) For speed $v = 2$ ft/sec the answer is $v \cdot D_{\hat{u}} f(1,1)$ where $\hat{u} = \langle 0,1 \rangle$ for positive y direction along $x = 1$.

$$\text{ANS} = v\, \vec{\nabla} f(1,1) \cdot \langle 0,1 \rangle = v f_y(1,1) = v \left(\frac{-200xy}{(x^2+y^2)^2} \right) \Bigg|_{(1,1)} = -100 \frac{{}^\circ F}{\text{sec}}$$

e) For $f(x,y) = F(r)$

$$f_x = F'(r) \frac{\partial}{\partial x} \sqrt{x^2+y^2} = F'(r) \frac{x}{\sqrt{x^2+y^2}} = F'(r) \frac{x}{r}$$

Similarly, $f_y = F'(r) \cdot y/r$ so that

$$\vec{\nabla} f = \langle f_x, f_y \rangle = F'(r) \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle = F'(r) \cdot \frac{1}{r} \vec{r}$$

By defn the unit radial vector $\hat{r} = \frac{1}{r} \vec{r}$ so

$$\vec{\nabla} f = F'(r)\, \hat{r}$$

(5) GEOMETRY

a) $\vec{N} = \langle -f_x(1,1), -f_y(1,1), 1 \rangle = \langle -2, -3, 1 \rangle$ $\qquad |\vec{N}| = \sqrt{14}$

Two unit vectors are $\pm \hat{N} = \pm \frac{1}{\sqrt{14}} \langle -2, -3, 1 \rangle$

b) $\vec{T}_1 = \langle f_x(1,1), 0, 1 \rangle = \langle 1, 0, 1 \rangle$

$\vec{T}_2 = \langle 0, f_y(1,1), 1 \rangle = \langle 0, 3, 1 \rangle$

c) Tangent Plane

$$\begin{aligned} f(x,y) &= x + y + xy^3 & f(1,1) &= 3 \\ f_x(x,y) &= 1 + y^3 & f_x(1,1) &= 2 \\ f_y(x,y) &= 1 + 3xy^2 & f_y(1,1) &= 4 \end{aligned}$$

Normal $\vec{N} = \langle -2, -4, 1 \rangle$ and $\vec{r}_0 = \langle 1, 1, 3 \rangle$ on plane

$$2x + 4y - z = 3$$

d) Normal $\qquad \vec{N} = \langle -f_x(1,1), -f_y(1,1), 1 \rangle = \langle -\frac{1}{4}, -\frac{3}{4}, 1 \rangle$

Thru $\vec{r}_0 = \langle x_0, y_0, f(x_0, y_0) \rangle = \langle 1, 1, 2 \rangle$ yields

$\vec{r}(t) = \langle 1, 1, 2 \rangle + t \langle -\frac{1}{4}, -\frac{3}{4}, 1 \rangle$ $\qquad$ Normal Line.

e) Normal Vector $\vec{N} = \langle -f_x(1,1), -f_y(1,1), 1 \rangle = \langle -\frac{1}{16}, -\frac{1}{8}, 1 \rangle$

Thru $\vec{r}_0 = \langle 1, 1, f(1,1) \rangle = \langle 1, 1, \frac{1}{4} \rangle$ yields

$\vec{r}(t) = \langle 1 - \frac{1}{16}t, 1 - \frac{1}{8}t, \frac{1}{4} + t \rangle$ $\qquad$ Normal Line L

intersects xy-plane when $z(t) = \frac{1}{4} + t = 0 \Leftrightarrow t = -\frac{1}{4}$

Intersection Pt $\vec{r}(-\frac{1}{4}) = \langle \frac{65}{64}, \frac{33}{32}, 0 \rangle$

f) $\vec{\nabla} f(x,y) = \langle 2xy, x^2 \rangle$ $\quad$ $\vec{\nabla} f(1,1) = \langle 2, 1 \rangle$ $\perp$ curve $x^2 y = 1$ @ $(1,1)$.

g) Level surface $g(x,y,z) = x^2 - z^3 + yz = 0$

$\vec{\nabla} g(x,y,z) = \langle 2x, z, y - 3z^2 \rangle$ $\perp$ surface

when $(x,y) = (1,0)$, $g(1,0,z) = 1 - z^3 = 0 \iff z = 1$
so $(x,y,z) = (1,0,1)$ on surface

$\vec{\nabla} g(1,0,1) = \langle 2, 1, -3 \rangle$ $\quad$ $\perp$ surface @ $(1,0,1)$.

h) $\vec{\nabla} f(x,y) = \langle 2x, 3y^2 \rangle$. $\vec{\nabla} f(1,1) = \langle 2, 3 \rangle$ is $\perp$ at $(1,1)$ but not unit length.

$|\vec{\nabla} f(1,1)| = \sqrt{13}$ $\quad$ $\hat{N} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle$ $\perp$

i) Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be the curve of intersection. Since $z = f(x,y)$ on such a curve, 'one' parametrization is $(x(t)^2 + y(t)^2 = 1)$

$\vec{r}(t) = \langle \cos t, \sin t, f(x(t), y(t)) \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, f_x \cdot \frac{dx}{dt} + f_y \frac{dy}{dt} \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, (x+y)(-\sin t) + (x-y)\cos t \rangle$

when $(x,y) = (1,0)$, $t = 0$ so tangent vector

$\vec{r}\,'(0) = \langle 0, 1, 1 \rangle$

by evaluating $\vec{r}\,'(t)$ above.

(6) CRITICAL POINTS AND 2ND DERIVATIVE TEST

a(i) $f = x^2 + 2y^2 - 4x + 4y$

$$\left.\begin{array}{l} f_x = 2x - 4 = 0 \\ f_y = 4y + 4 = 0 \end{array}\right\} \quad \begin{array}{l}(x,y) = (2,-1) \\ \text{sole ct. pt.}\end{array}$$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (2)(4) - 0^2 = 8 > 0$ and $f_{xx} = 2 > 0$ hence $(2,-1)$ is a local min.

a(iv) $f = x^4 + y^4 - 4xy$

(1) $f_x = 4x^3 - 4y = 0$

(2) $f_y = 4y^3 - 4x = 0$

Eqn true only if $y = x^3$. Using this in (2)

$$x^9 - x = x(x^8 - 1) = 0 \qquad x = 0, \pm 1$$

Yields 3 ct. pts. $(0,0)$ $(1,1)$ and $(-1,-1)$

(x,y)	$D = 144x^2y^2 - 16$	$f_{xx} = 12x^2$	Conclude
$(0,0)$	$-$	$+$	saddle
$(1,1)$	$+$	$+$	loc. min
$(-1,-1)$	$+$	$+$	loc. min

b) Since $\nabla f = \langle x^2 - y, y - x \rangle$ then

(1) $f_x = x^2 - y = 0$
(2) $f_y = y - x = 0$

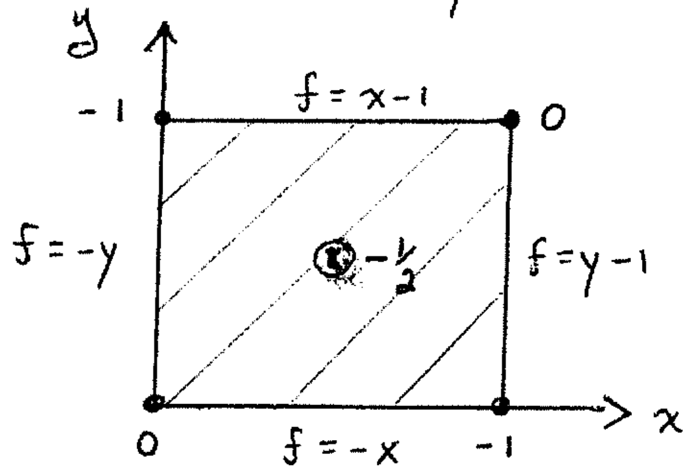
yields $x^2 - x = x(x-1) = 0$ so ct pts $(0,0)$ and $(1,1)$.
From (1), $f_{xx} = 2x$, $f_{xy} = -1$. From (2), $f_{yy} = 1$.
Conclude $D = 2x - 1$ and classifying

$(0,0)$ saddle $\qquad$ $(1,1)$ local min

(7) MAXIMA AND MINIMA ON A REGION

a) The function $f = 2xy - x - y$ has the sole critical point $(x_0, y_0) = (\tfrac{1}{2}, \tfrac{1}{2})$ and it is inside the unit square.

⊗ ct pt



$f(0,y) = -y$

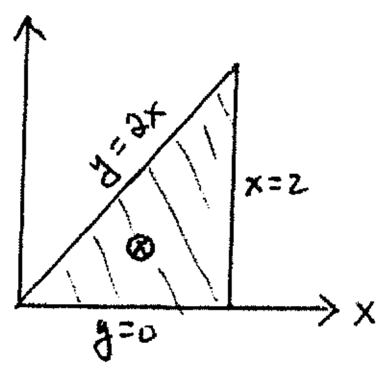
$f(x,1) = x-1$

$f(1,y) = y-1$

$f(x,0) = -x$

Since $f(\tfrac{1}{2},\tfrac{1}{2}) = -\tfrac{1}{2}$ the max f is 0 and it occurs at $(0,0)$, $(1,1)$ on boundary.

b) $f = x^2 - x + y - xy$ has one ct. pt. at $(x,y) = (1,1)$.



$f(x,0) = x^2 - x = (x-\tfrac{1}{2})^2 - \tfrac{1}{4}$

$f(2,y) = 2 - y$

$f(x,2x) = x - x^2 = -(x-\tfrac{1}{2})^2 + \tfrac{1}{4}$

are values of f on 3 edges

$\max_{x\in[0,2]} f(x,0) = f(2,0) = 2$] $\min_{x\in[0,2]} f(x,0) = f(\tfrac{1}{2},0) = -\tfrac{1}{4}$

$\max_{y\in[0,4]} f(2,y) = f(2,0) = 2$] $\min_{y\in[0,4]} f(2,y) = f(2,4) = -2$

$\max_{x\in[0,2]} f(x,2x) = f(\tfrac{1}{2},1) = \tfrac{1}{4}$ $\min_{x\in[0,2]} f(x,2x) = f(2,4) = -2$

Given, at ct. pt, $f(1,1) = 0$ then abs min/max;

abs max $= +2$ at $(2,0)$]

abs min $= -2$ at $(2,4)$ ○

(8) LAGRANGE MULTIPLIERS

a)

$$(1) \quad 1 = 3x^2\lambda \qquad\qquad f_x = \lambda g_x$$
$$(2) \quad 1 = 3y^2\lambda \qquad\qquad f_y = \lambda g_y$$
$$(3) \quad x^3 + y^3 = 16 \qquad\qquad g = 0$$

Divide (2) and (1) since $x, y, \lambda \neq 0$. $y^2 = x^2$ hence $y = \pm x$.

$y = -x$ in (3) yields the contradiction $0 = 16$. Thus $y = +x$ only which in (3) yields $2x^3 = 16$ or $x = 2$.

$$(x, y) = (2, 2) \qquad \text{sole extrema.}$$

b) Problem has many many extrema.

$$(1) \quad 2x = 4\lambda x^3$$
$$(2) \quad 2y = 4\lambda y^3$$
$$(3) \quad x^4 + y^4 = 16$$

$x = 0$ in (1) and (3) $\Rightarrow y = \pm 2$ $\qquad (x, y, \lambda) = (0, \pm 2, \tfrac{1}{8})$

$y = 0$ in (2) and (3) $\Rightarrow x = \pm 2$ $\qquad (x, y, \lambda) = (\pm 2, 0, \tfrac{1}{8})$

If both $x, y \neq 0$ then dividing (2) and (1) implies $y = \pm x$. Using $y = \pm x$ in (3) one finds

$$2x^4 = 16 \qquad\qquad x = \pm\sqrt[4]{8}$$

Thus we have (in totality) 8 extrema

$$(x, y) = (0, \pm 2), \ (\pm 2, 0)$$
$$(x, y) = \pm(\sqrt{8}, \sqrt[4]{8}), \ \pm(\sqrt[4]{8}, -\sqrt[4]{8})$$

c) Since f has no critical points the max/min must occur on the boundary. Must solve (for $f = x + 2y$)

$$\max f(x,y) \qquad g(x,y) = x^2 + y^2 = 5$$

Lagrange Multiplier λ.

$$\begin{aligned} (1) \quad & 1 = 2\lambda x \\ (2) \quad & 2 = 2\lambda y \\ (3) \quad & x^2 + y^2 = 5 \end{aligned}$$

Can't have $x, y = 0$ in (1) or (2) so both $x, y \neq 0$. Thus divide (1) and (2)

$$y = 2x$$

which used in (3) yields $5x^2 = 5$ or $x = \pm 1$

$$\begin{aligned} (x, y) &= (1, 2) \qquad f = 5 \qquad \text{abs max} \\ (x, y) &= (-1, -2) \qquad f = -5 \qquad \text{abs min.} \end{aligned}$$

d) Must find extrema on boundary and compare to value $f(0,0) = 0$ at sole critical point $(0,0)$ inside region

$$\left.\begin{aligned} (1) \quad & 2x + y = 2\lambda x \\ (2) \quad & x + 2y = 2\lambda y \\ (3) \quad & x^2 + y^2 = 1 \end{aligned}\right\} \text{ divide and simplify to get } y^2 = x^2$$

Eqns (1)-(2) imply $y = \pm x$ which used in (3) yields $2x^2 = 1$, $x = \pm \frac{1}{\sqrt{2}}$ or 4 points

	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$f =$	$3/2$ abs max	$1/2$	$\frac{1}{2}$	$\frac{3}{2}$ abs max

Ct Pt is abs min since $f(0,0) = 0$ smallest.

e) Sufficient to minimize distance squared $f = x^2 + y^2$ subject to constraint $g = xy - 4 = 0$

(1) $2x = \lambda y$
(2) $2y = \lambda x$
(3) $xy = 4$

Clearly neither $x, y = 0$ else (3) not satisfied. Thus divide (1) and (2) to get (after simplifying)

$$y = \pm x \qquad (y^2 = x^2)$$

which used in (3) yields $x^2 = 4$, $x = \pm 2$

The minimum distance is $\sqrt{2^2 + 2^2} = \sqrt{8}$ and occurs at $\pm(2, 2)$ points.

f) Minimize $f = x^2 + y^2 + z^2$ subject $g = x + 2y - 3z = 1$

(1) $2x = \lambda \Rightarrow x = \lambda/2$
(2) $2y = 2\lambda \Rightarrow y = \lambda$
(3) $2z = -3\lambda \Rightarrow z = -\frac{3}{2}\lambda$
(4) $x + 2y - 3z = 1$

using $x = \frac{1}{2}\lambda$, $y = \lambda$, $z = -\frac{3}{2}\lambda$ in Eqn (4) yields

$$7\lambda = 1$$

hence $\lambda = \frac{1}{7}$ and

$$(x, y, z) = \left(\frac{1}{14}, \frac{1}{7}, -\frac{3}{14}\right)$$

is the point on the plane closest to $(0, 0, 0)$. The minimum distance is

$$D_{min} = \sqrt{\left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2} = \frac{1}{\sqrt{14}}$$

g) Minimize $f = x + y - z$ sub to $g = x^2 + y^2 + z^2 = 1$

$$\left.\begin{array}{lrcl} (1) & 1 & = & 2\lambda x \\ (2) & 1 & = & 2\lambda y \\ (3) & -1 & = & 2\lambda z \\ (4) & \multicolumn{3}{l}{x^2 + y^2 + z^2 = 1} \end{array}\right\} \text{ solve } x, y, z \text{ in terms of } \lambda.$$

Find $x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ and $z = -\frac{1}{2\lambda}$. Use in (4) to find

$$\lambda = \pm \frac{\sqrt{3}}{2} \qquad \left(\lambda^2 = \frac{3}{4}\right)$$

Thus obtain two points (extrema)

$$(x, y, z) = \pm \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \pm \vec{X}$$

Evaluating at these pts "+" makes f a max hence min is $f(-\vec{X}) = -\sqrt{3}$ at $-\vec{X}$.

h) Base x, height y. Cost $C(x,y) = 2x^2 + 4xy$. Minimize cost $C(x,y)$ subject to fixed volume $g = x^2 y = V$.

$$\left.\begin{array}{l} C_x = \lambda g_x \\ C_y = \lambda g_y \end{array}\right\} \text{ divide yields } y = x, \text{ i.e. base the same as height (or a cube!)}$$

i) Radius r, height h. Note liter $= 10^3 \text{cm}^3$.

$$S = 2\pi r h + \pi r^2 \qquad \text{(surface area)}$$
$$V = \pi r^2 h \qquad \text{(Volume)}$$

Minimize S subject to $V = 1$

$$\left.\begin{array}{lrcl} (1) & S_r & = & \lambda V_r \\ (2) & S_h & = & \lambda V_h \\ (3) & V & = & 1000 \text{ cm}^3 \end{array}\right\} \quad r = h = \frac{10}{\sqrt[3]{\pi}} \text{ cm}$$