Math 450 (2011) – Final (Take home)

Due: Wednesday, December 14, 2011 (11am) NAME: ________________

Get the exam no me no later than 11am on December 14, 2011. You may give me the exam in person at my office (Wil 2-136) or slide it under my office door. You may use the text, posted notes or your class notes only. You may not work with anyone but you can ask me to clarify questions.

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1. [25 pts] Let $y(t,\epsilon)$ be the solution of the initial value problem

$$\begin{aligned} y'' + y &= \epsilon y\,(y')^4 \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 0 \quad , \quad y'(0) = 1 + 2\epsilon \end{aligned}$$

where $(\ )'$ denotes differentiation in t. Assume

$$\begin{aligned} y(t,\epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon)t \equiv (1 + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots)t \end{aligned}$$

a) Use Poincare-Lindstedt's method to determine ω_1 and the $O(\epsilon)$ correction to the period of the oscillation. You may use the identity:

$$\sin A \cos^4 A = \frac{1}{8}\sin A + \frac{3}{16}\sin 3A + \frac{1}{16}\sin 5A$$

b) What initial conditions must $y_1(\tau)$ satisfy? Do not compute $y_1(\tau)$.

2. [25 pts] The following equation has three real roots for small positive ϵ.

$$\epsilon x^2 + \frac{2}{1+x} = 1 \quad , \quad 0 < \epsilon \ll 1 \tag{1}$$

Find a two term expansions for the singular roots:

$$x = \frac{X}{\delta} = \frac{1}{\delta}\left(X_0 + \delta X_1 + O(\delta^2)\right) \qquad , \qquad 0 < \delta \ll 1$$

When finding δ make sure you balance the two largest of (1) terms noting that

$$\frac{1}{1+x} = \frac{\delta}{\delta + X} \ll 1$$

If you're having difficulty finding X_1, at least state the two values for X_0.

3. [25pts] Let $y(x, \epsilon)$ be the solution of the following boundary value problem:

$$\begin{aligned} \epsilon x^{1/2} y'' + x^{1/2} y' + y^2 &= 0 \quad , \quad x \in (0,1) \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 1 \quad , \quad y(1) = 2 \end{aligned}$$

a) Find the leading order outer approximation $y_0(x)$ satisfying the right boundary condition $y_0(1) = 2$. The equation is separable.

b) Find the boundary layer thickness $\delta(\epsilon)$ and $Y_0(X)$ in the inner expansion

$$y(x, \epsilon) = Y(X, \epsilon) = Y_0(X) + o(1) \qquad , \qquad X \equiv \frac{x}{\delta}$$

that satifies $Y_0(0) = 1$

c) Find a uniformly valid approximation $y_u(x, \epsilon)$ of the solution for the problem.

4. [25 pts] A functional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$\begin{aligned} J(y) &= \int_0^{\pi} L(y(x), y'(x)) \, dx \\ \mathcal{A} &= \{y \in C^2[0, \pi] : y(0) = 1, y(\pi) = \frac{1}{2}\} \end{aligned}$$

where the Lagrangian

$$L(y, y') = -\frac{(y')^2}{1 - y^2}$$

Use the first integral

$$L - y' L_{y'} = c \qquad , \qquad c \in \mathbb{R}$$

of the Euler-Lagrange equations to find <u>all</u> the extrema $\bar{y} \in \mathcal{A}$ of the functional J. You may assume $1 - \bar{y}^2 > 0$ and $\bar{y}' > 0$. The first integral above is a separable differential equation for $y(x)$ where c is a constant of integration. One of the boundary conditions is satisfied for many c values owing to the periodicity of the solution.