QUESTION ONE

$$(1) \quad y'' + y = \varepsilon y (y')^4$$

$$(2) \quad y(0) = 0 \quad , \quad y'(0) = 1 + 2\varepsilon$$

Expansion

$$y(t, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2)$$

$$\tau = (1 + \omega_1 \varepsilon + \cdots) t = \omega(\varepsilon) t$$

General theory in notes imply

$$y'(t) = y_0'(\tau) + (y_1'(\tau) + \omega_1 y_0'(\tau))\varepsilon + O(\varepsilon^2)$$

$$y''(t) = y_0''(\tau) + (y_1''(\tau) + 2\omega_1 y_0''(\tau))\varepsilon + O(\varepsilon^2)$$

Thus y_0 and y_1 must satisfy

$$(3) \quad y_0'' + y_0 = 0 \qquad O(1)$$

$$(4) \quad y_1'' + y_1 = y_0 (y_0')^4 - 2\omega_1 y_0'' \qquad O(\varepsilon)$$

Given the B. Conds

$$y_0(0) = 0 \qquad y_0'(0) = 1$$

it is clear

$$(5) \quad y_0(\tau) = \sin \tau$$

To find ω_1 we use (5) in (4) and identify secular terms.

$$y_1'' + y_1 = \sin\tau\cos^4\tau + 2\omega_1\sin\tau$$

$$y_1'' + y_1 = \underbrace{\left(\tfrac{1}{8} + 2\omega_1\right)}_{\substack{\text{must vanish to} \\ \text{assure } y_1 \text{ is } 2\pi\text{-period}}}\sin\tau + \frac{3}{16}\sin 3\tau + \frac{1}{16}\sin 5\tau$$

Conclude

$$\boxed{\omega_1 = -\frac{1}{16}}$$

and the period correction is

$$T = \frac{2\pi}{\omega(\varepsilon)} = 2\pi\left(1 + \frac{1}{16}\varepsilon + O(\varepsilon^2)\right)$$

Initial Condition for $y_1(\tau)$

$$y'(t,\varepsilon) = y_0'(\tau) + (y_1'(\tau) + \omega_1 y_0'(\tau))\varepsilon + \cdots$$

So at $\varepsilon = 0$

$$1 \underset{\Uparrow}{+ 2\varepsilon} = y_0'(0) + (y_1'(0) \underset{\Uparrow}{+} \omega_1 y_0'(0))\varepsilon + \cdots$$

Indicated terms must be equal

$$y_1'(0) = 2 - \omega_1 y_0'(0)$$

$$y_1'(0) = 2 - \left(-\frac{1}{16}\right)(1)$$

$$y_1'(0) = \frac{33}{16}$$

QUESTION TWO Singular roots of

$$(1) \qquad \varepsilon x^2 + \frac{2}{1+x} = 1$$

First let

$$x = \frac{X}{\delta} \qquad 0 < \delta \ll 1$$

then (1) becomes

$$(2) \qquad \frac{\varepsilon}{\delta^2} X^2 - 1 + \frac{2\delta}{\delta + X} = 0$$

① ~ ② >> ③ (for any $\delta \ll 1$)

Dominant balance yields

$$\delta(\varepsilon) = \varepsilon^{1/2}$$

and (2) is regular in δ:

$$(3) \qquad X^2 - 1 + \frac{2\delta}{\delta + X} = 0$$

Now let

$$X = X_0 + \delta X_1 + O(\delta^2)$$

The last term in (3) is easy to expand:

$$F(\delta) = \delta G(\delta) = \delta \, (G(0) + G'(0)\delta + \cdots)$$

where $G(\delta) = 2(\delta + X_0 + \delta X_1 + \cdots)^{-1}$. Hence

$$\frac{2\delta}{\delta + X_0} = \frac{2}{X_0}\delta + O(\delta^2)$$

Thus (3) becomes

$$(X_0 + \delta X_1 + \cdots)^2 - 1 + \frac{2\delta}{X_0} + O(\delta^2) = 0$$

Collecting like powers of δ.

$$X_0^2 - 1 = 0$$

$$2 X_0 X_1 + \frac{2}{X_0} = 0$$

Clearly $X_0 = \pm 1$. For both $X_1 = -1$. Conclude

$$x_+ = \frac{1}{\varepsilon^{1/2}}(1 - \varepsilon^{1/2} + o(\delta)) = \frac{1}{\varepsilon^{1/2}} - 1 + o(1)$$

$$x_- = \frac{1}{\varepsilon^{1/2}}(-1 - \varepsilon^{1/2} + o(\delta)) = -\frac{1}{\varepsilon^{1/2}} - 1 + o(1)$$

i.e,

$$X = \pm 1 - \varepsilon^{1/2} + o(\varepsilon^{1/2})$$

QUESTION THREE

(1) $$\varepsilon x^{1/2} y'' + x^{1/2} y' + y^2 = 0$$

(2) $$y(0) = 1 \qquad y(1) = 2$$

a) **OUTER EXPANSION** $y(x,\varepsilon) = y_0(x) + o(1)$

(3) $$x^{1/2} y_0' + y_0^2 = 0$$

(4) $$y_0(1) = 2$$

has solution (general)

$$y_0(x) = \frac{1}{2\sqrt{x} + C} \qquad C \in \mathbb{R}$$

To satisfy $y_0(1)$

$$\boxed{y_0(x) = \frac{2}{4\sqrt{x} - 3}} \qquad \left(x \neq \frac{9}{16}\right)$$

b) **INNER EXPANSION**

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\delta}, \qquad \delta \ll 1$$

Equation (1) becomes

(5) $$\frac{\varepsilon}{\delta^{3/2}} X^{1/2} Y'' + \frac{1}{\delta^{1/2}} X^{1/2} Y' + Y^2 = 0$$

$$① \sim ② \gg ③$$

Dominant Balance

$$\delta(\varepsilon) = \varepsilon$$

Hence (5) Becomes

$$\bar{X}^{1/2}\bar{Y}'' + \bar{X}^{1/2}\bar{Y}' + \varepsilon^{1/2}\bar{Y}^2 = 0$$

Then $\bar{Y}(\bar{X},\varepsilon) = \bar{Y}_0(\bar{X}) + o(1)$ implies

$$(6) \qquad \bar{Y}_0'' + \bar{Y}_0' = 0$$

$$(7) \qquad \bar{Y}_0(0) = 1$$

Solution of (6)-(7) is

$$\boxed{\bar{Y}_0(\bar{X}) = 1 + A(e^{-\bar{X}} - 1)} \qquad A \in \mathbb{R}$$

c) <u>Uniform Solution: Must Match</u>

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{\bar{X} \to \infty} \bar{Y}_0(\bar{X})$$

$$M = -\tfrac{2}{3} = 1 - A$$

Hence $A = \tfrac{5}{3}$ and

$$\bar{Y}_0(\bar{X}) = -\tfrac{2}{3} + \tfrac{5}{3}e^{-\bar{X}}$$

Uniform solution

$$y_u(x,\varepsilon) = y_0(x) + \bar{Y}_0\left(\frac{x}{\varepsilon}\right) - M$$

$$y_u(x,\varepsilon) = \frac{2}{4\sqrt{x} - 3} + \tfrac{5}{3}e^{-x/\varepsilon}$$

Interior Layer at $y_0(x)$ singularity: Complicated.

$$y(x) = \tilde{Y}(z) \qquad z = \frac{x - x_0}{\delta}, \quad y = \frac{Y}{\delta}$$

QUESTION FOUR

$$\boxed{L(y,y') = \frac{-(y')^2}{1-y^2}}$$

First integral. Calculations show

$$L - y' L_{y'} = \frac{(y')^2}{1-y^2} = c_1^2 \qquad c_1 \in \mathbb{R}$$

which is separable

$$\frac{dy}{\sqrt{1-y^2}} = c_1\, dx$$

General solution

$$y(x) = \sin(c_1 x + c_2)$$

Extrema satisfy $y(0) = 1$ and $y(\pi) = \frac{1}{2}$.
First boundary condition

$$y(0) = \sin(c_2) = 1 \qquad c_2 = \frac{\pi}{2} + 2n\pi$$

so that

$$y(x) = \sin\left(c_1 x + \frac{\pi}{2} + 2n\pi\right) = \sin\left(c_1 x + \frac{\pi}{2}\right)$$

$$y(x) = \cos(c_1 x)$$

Second boundary condition

$$y(\pi) = \cos(c_1 \pi) = \frac{1}{2} \qquad c_1 \pi = \pm\frac{\pi}{3} + 2m\pi$$

Hence

$$c_1 = \pm\frac{1}{3} + 2m$$

$$\bar{y}(x) = \cos\left(2mx \pm \frac{1}{3}x\right) \qquad \text{not unique.}$$