

## Math 450 (2011) – Midterm 1 (Take home)

Due: October 28, 2011.

NAME: \_\_\_\_\_

**Instructions:** You may not talk to anyone except me. You can not use any computer software such as Matlab, Maple or Mathematica. You may use your class notes, the textbook and any materials posted on the class webpage. You may not use the internet otherwise.

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1. [30pts] A fluid of density  $\rho$  flows down a pipe of diameter  $r$  with a mean velocity  $v$ . The viscosity  $\mu$  of the fluid has dimensions  $[\mu] = ML^{-1}T^{-1}$ . The friction of the flow against the pipe causes a pressure drop  $P$  over the length of the pipe. Pressure is a measure of force per unit area. We shall assume these quantities are related as follows:

$$f(\rho, r, v, \mu, P) = 0$$

Use dimensional analysis to find a formula for  $P$  in terms of the other dimensional quantities. Use  $M, L, T$  as fundamental units and the following convention for dimensionless groups:

$$\Pi = \rho^{\alpha_1} r^{\alpha_2} v^{\alpha_3} \mu^{\alpha_4} P^{\alpha_5}$$

2. [20pts] A model for an enzyme-substrate system is given by

$$\begin{aligned} \frac{dS}{dT} &= -k_1 E_0 S + (k_1 S + k_2) C & , & & S(0) = S_0 \\ \frac{dC}{dT} &= k_1 E_0 S - (k_1 S + k_3) C & , & & C(0) = 0 \end{aligned}$$

Here  $S$  and  $C$  are the concentrations ( $ML^{-3}$ ) of the substrate and substrate-enzyme complex, respectively.  $E_0$  is the initial dimensional concentration of the enzyme,  $T$  is dimensional time and  $k_1, k_2, k_3$  are constants. Show the model has a dimensionless version:

$$\begin{aligned} \frac{ds}{dt} &= -s + (s + \alpha)c & , & & s(0) = 1 \\ \frac{dc}{dt} &= \gamma(s - (s + \beta)c) & , & & c(0) = 0 \end{aligned}$$

where  $s, c, t, \alpha, \beta, \gamma$  are dimensionless. Express the latter three parameters in terms of dimensional quantities.

3. [25pts] The equation

$$f(x, \epsilon) = x - \log(1 + \epsilon x) - 2\sqrt{1 + \epsilon x} = 0 \quad , \quad 0 < \epsilon \ll 1$$

has a single regular root  $\bar{x}(\epsilon)$  with the expansion

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3)$$

Determine  $x_0, x_1$  and  $x_2$  using the following small  $z$  expansions:

$$\begin{aligned} \log(1 + z) &= z - \frac{1}{2}z^2 + O(z^3) \\ \sqrt{1 + z} &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + O(z^3) \end{aligned}$$

4. [25pts] Consider the perturbed first order initial value problem:

$$\frac{dy}{dx} = e^{-y} + \epsilon y \quad , \quad y(1) = 0$$

where  $0 < \epsilon \ll 1$ . Find  $y_0(x)$  and  $y_1(x)$  in the assumed expansion of the solution  $y$ :

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

Here the leading  $O(1)$  problem for  $y_0(x)$  is a separable. Note the initial value of  $x$  is  $x = 1$ .

The problem for  $y_1(x)$  is linear and you may use integral tables when seeking its solution via integrating factors. The answer for  $y_0(x)$  is simple. In fact  $e^{-y_0}$  is very simple.

### QUESTION ONE

$$[\rho] = ML^{-3} \quad [r] = L \quad [v] = LT^{-1}$$

$$[\mu] = ML^{-1}T^{-1} \quad [P] = ML^{-1}T^{-2}$$

Hence dimensionless groups are

$$\Pi = (ML^{-3})^{d_1} (L)^{d_2} (LT^{-1})^{d_3} (ML^{-1}T^{-1})^{d_4} (ML^{-1}T^{-2})^{d_5}$$

yields a dimension matrix  $A$  for  $\vec{d} = (d_1, d_2, d_3, d_4, d_5)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ -3 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & \downarrow & \downarrow \\ 0 & \textcircled{1} & 1 & 2 & 2 \\ 0 & 0 & \textcircled{1} & 1 & 2 \end{bmatrix} \quad d_4, d_5 \text{ free}$$

Letting  $(d_4, d_5) = (0, 1)$  and  $(d_4, d_5) = (1, 0)$  backsolve

$$\vec{d}_1 = (-1, -1, -1, 1, 0)$$

$$\Pi_1 = \frac{\mu}{\rho r v}$$

$$\vec{d}_2 = (-1, 0, -2, 0, 1)$$

$$\Pi_2 = \frac{P}{\rho v^2}$$

yields

$$\boxed{P = \rho v^2 f\left(\frac{\mu}{\rho r v}\right)}$$

for some function  $f$ .

## QUESTION TWO

$$S' = s S'^* \quad C' = c C'^* \quad \pi = t \pi^*$$

where constants  $S'^*$ ,  $C'^*$ ,  $\pi^*$  to be determined.

$$(1) \quad \frac{ds}{dt} = - \underbrace{(k_1 E_0 \pi^*)}_1 s + \left\{ \underbrace{(k_1 \pi^* C'^*)}_1 s + \underbrace{\frac{k_2 \pi^* C'^*}{S'^*}}_\alpha \right\} c$$

Since  $S'(0) = S'_0$  we must choose  $S'^* = S'_0$ . The requirement the braced terms equal one  $\Rightarrow$

$$(2) \quad S'^* = S'_0 \quad C'^* = E_0 \quad \pi^* = \frac{1}{k_1 E_0} \quad \alpha = \frac{k_2}{k_1 S'_0}$$

Then we find

$$(3) \quad \frac{dc}{dt} = \frac{\pi^*}{C'^*} \left\{ \underbrace{(k_1 E_0 S'^*)}_\uparrow s - (k_1 S'^* s + k_3) c \right\}$$

After using (2) in (3) we note the coefficient of the indicated term is  $\gamma$ .

$$(4) \quad \gamma = \frac{k_1 E_0 S'^* \pi^*}{C'^*} = \frac{S'_0}{E_0}$$

Then (3) becomes

$$\frac{dc}{dt} = \gamma (s - (s + \beta) c) \quad \beta = \frac{k_3}{k_1 S'_0}$$

$$\frac{ds}{dt} = -s + (s + \alpha) c \quad \alpha = \frac{k_2}{k_1 S'_0}$$

### QUESTION THREE

$$(1) \quad \bar{x}(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3)$$

Expand  $\log(1 + \epsilon \bar{x})$

$$\log(1 + \epsilon \bar{x}) = \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) - \frac{1}{2}(x_0 + \epsilon x_1 + \dots)^2 \epsilon^2 + O(\epsilon^3)$$

$$(2) \quad \log(1 + \epsilon \bar{x}) = \epsilon(x_0) + \epsilon^2(x_1 - \frac{1}{2}x_0^2) + O(\epsilon^3)$$

Expand  $2\sqrt{1 + \epsilon \bar{x}}$

$$2\sqrt{1 + \epsilon \bar{x}} = 2 + \epsilon(x_0 + \epsilon x_1 + \dots) - \frac{1}{4}(x_0 + \epsilon x_1 + \dots)^2 \epsilon^2 + O(\epsilon^3)$$

$$(3) \quad 2\sqrt{1 + \epsilon \bar{x}} = 2 + \epsilon x_0 + (x_1 - \frac{1}{4}x_0^2)\epsilon^2 + O(\epsilon^3)$$

Using (1)-(3) in  $f(x, \epsilon) = x - \log(1 + \epsilon x) - 2\sqrt{1 + \epsilon x}$  and collecting powers yields the problems

$$x_0 - 2 = 0 \quad O(1)$$

$$x_1 - 2x_0 = 0 \quad O(\epsilon)$$

$$x_2 - 2x_1 + \frac{3}{4}x_0^2 = 0 \quad O(\epsilon^2)$$

whose soln is

$$x_0 = 2 \quad x_1 = 4 \quad x_2 = 5$$

i.e.,

$$\bar{x}(\epsilon) = 2 + 4\epsilon + 5\epsilon^2 + O(\epsilon^3)$$

QUESTION FOUR Need the following expansion

$$\begin{aligned}e^{-y} &= e^{-y_0} e^{-\varepsilon y_1} e^{-\varepsilon^2 y_2} \dots \\ &= e^{-y_0} (1 - (\varepsilon y_1 + \varepsilon^2 y_2 + \dots)) + O(\varepsilon^2) \\ &= e^{-y_0} (1 - \varepsilon y_1 + O(\varepsilon^2))\end{aligned}$$

Note: don't need  $O(\varepsilon^2)$  terms. Then expanding

$$y' = e^{-y} + \varepsilon y \quad y(1) = 0$$

we get

$$O(1) \quad y_0' = e^{-y_0}, \quad y_0(1) = 0$$

$$O(\varepsilon) \quad y_1' = -y_1 e^{-y_0} + y_0, \quad y_1(1) = 0$$

The solution of the  $O(1)$  problem is

$$y_0(x) = \ln x$$

Noting  $e^{-y_0} = \frac{1}{x}$  the  $O(\varepsilon)$  problem is linear

$$y_1' + \frac{1}{x} y_1 = \ln x \quad y_1(1) = 0$$

Solving, ...

$$y_1(x) = \frac{1}{2} x \ln x - \frac{1}{4} x + \frac{1}{4x}$$