

Math 450 (2009) – Midterm 1 (Take home)

Due: October 23, 2009.

NAME: _____

1. [30pts] A fluid of density ρ exerts a (drag) force F on a cylinder of diameter D as it flows around it. The fluid has viscosity μ and the fluid velocity far from the cylinder is v . Assume the physical law

$$f(\rho, F, D, \mu, v) = 0$$

and then find all the dimensionless π of the form

$$\pi = \rho^{\alpha_1} F^{\alpha_2} D^{\alpha_3} \mu^{\alpha_4} v^{\alpha_5}$$

Note that the units of viscosity are $[\mu] = ML^{-1}T^{-1}$.

2. [20pts] In quantum mechanics the wave function $\psi(X)$ of a particle of mass m is a solution of Schrödinger's equation. For the quantum harmonic oscillator problem, the (time-independent) Schrödinger equation is:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dX^2} + \frac{1}{2}m\omega^2 X^2\psi = E\psi$$

where X is position, ω is the frequency of the oscillator (potential well), \hbar is Planck's constant and E is the energy of the system. The wave function ψ depends on position X but since it is a probability density function (used to determine the location of the particle) it has no units.

Note here that $[\omega] = T^{-1}$ where T is time.

i) Determine the units of \hbar in terms of M, T, L

ii) Nondimensionalize the problem so it has only one dimensionless quantity, i.e.

$$-\frac{d^2\psi}{dx^2} + (x^2 - \mathcal{E})\psi = 0$$

3. [25pts] The equation

$$f(x, \epsilon) = x - \frac{1}{(x + \epsilon)^3} = 0 \quad , \quad 0 < \epsilon \ll 1$$

has two real roots $\bar{x}_{\pm}(\epsilon)$. Assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

and then determine x_0 and x_1 for both roots. If you use the Binomial theorem note

$$\begin{aligned} (x + \epsilon)^p &= (x_0 + \epsilon x_1 + \cdots + \epsilon)^p \\ &= (x_0 + \epsilon(x_1 + 1) + O(\epsilon^2))^p \\ &= x_0^p \left(1 + \epsilon \frac{(x_1 + 1)}{x_0} + O(\epsilon^2) \right)^p \end{aligned}$$

Then you can expand since the latter term has the form $(1 + z)^p$.

4. [25pts] Consider the perturbed first order initial value problem:

$$\frac{dy}{dx} + y^2 = \epsilon \left(\frac{1}{y} - x \right) \quad , \quad y(0) = 1$$

where $0 < \epsilon \ll 1$. Find $y_0(x)$ and $y_1(x)$ in the assumed expansion of the solution y :

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

Here the leading $O(1)$ problem for $y_0(x)$ is a separable (and Bernoulli).

QUESTION ONE

$$[\rho] = ML^{-3} \quad \alpha_1$$

$$[F] = ML\pi^{-2} \quad \alpha_2$$

$$[D] = L \quad \alpha_3$$

$$[\mu] = ML^{-1}\pi^{-1} \quad \alpha_4$$

$$[v] = L\pi^{-1} \quad \alpha_5$$

For $\pi = \rho^{\alpha_1} F^{\alpha_2} D^{\alpha_3} \mu^{\alpha_4} v^{\alpha_5}$ we have

$$[\pi] = M^{(\alpha_1 + \alpha_2 + \alpha_4)} L^{(-3\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5)} \pi^{(-2\alpha_2 - \alpha_4 - \alpha_5)}$$

To be dimensionless $[\pi] = 1$ implies
 $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in N(A)$ where the
dimension matrix A is

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -3 & 1 & 1 & -1 & 1 \\ 0 & -2 & 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 4 & 1 & 2 & 1 \\ 0 & 0 & 1 & \textcircled{0} & \textcircled{-1} \end{bmatrix} \quad \alpha_4, \alpha_5 \text{ free}$$

$(\alpha_4, \alpha_5) = (0, 1)$ yields $\vec{\alpha} = (\frac{1}{2}, -\frac{1}{2}, 1, 0, 1)$

$$\pi_1 = Dv\sqrt{\frac{\rho}{F}}$$

$(\alpha_4, \alpha_5) = (1, 0)$ yields $\vec{\alpha} = (-\frac{1}{2}, -\frac{1}{2}, 0, 1, 0)$

$$\pi_2 = \frac{\mu}{\sqrt{\rho F}}$$

QUESTION THREE

$$f(x, \varepsilon) = x - \frac{1}{(x+\varepsilon)^3} = 0$$

Using $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$ we get (for $p = -3$)

$$(x_0 + \varepsilon x_1 + \dots) - (x_0 + \varepsilon x_1 + \dots + \varepsilon)^p = 0$$

$$(x_0 + \varepsilon x_1 + \dots) - x_0^{-3} \left(1 + \varepsilon \left(\frac{x_1+1}{x_0}\right) + \dots\right)^{-3} = 0$$

$$(x_0 + \varepsilon x_1 + \dots) - x_0^{-3} \left(1 - 3\varepsilon \left(\frac{x_1+1}{x_0}\right) + \dots\right) = 0$$

Collecting like powers of ε

$$O(1) \quad x_0 - \frac{1}{x_0^3} = 0$$

$$O(\varepsilon) \quad x_1 + \frac{3(x_1+1)}{x_0^4} = 0$$

Solutions of $O(1)$ problem are $x_0 = \pm 1$.

Regardless $O(\varepsilon)$ problem simplifies to

$$4x_1 + 3 = 0$$

so that $x_1 = -\frac{3}{4}$ and

$$\bar{x}_+ = 1 - \frac{3}{4}\varepsilon + O(\varepsilon^2)$$

$$\bar{x}_- = -1 - \frac{3}{4}\varepsilon + O(\varepsilon^2)$$

QUESTION FOUR

$$y' + y^2 = \varepsilon \left(\frac{1}{y} - x \right)$$

Let $y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$, then

$$(y_0' + \varepsilon y_1' + \dots) + (y_0 + \varepsilon y_1 + \dots)^2 = \varepsilon \left(\frac{1}{y_0 + \varepsilon y_1 + \dots} - x \right)$$

Retaining only $O(1)$ and $O(\varepsilon)$ terms

$$(y_0' + y_0^2) + \varepsilon(y_1' + 2y_0 y_1) = \varepsilon \left(\frac{1}{y_0} + O(\varepsilon) - x \right)$$

yields

$$O(1) \quad y_0' + y_0^2 = 0 \quad y_0(0) = 1$$

$$O(\varepsilon) \quad y_1' + 2y_0 y_1 = \frac{1}{y_0} - x \quad y_1(0) = 0$$

The $O(1)$ problem is Bernoulli / Separable:

$$y_0(x) = \frac{1}{1+x}$$

Then the $O(\varepsilon)$ problem reduces to

$$y_1' + \frac{2}{1+x} y_1 = 1, \quad y_1(0) = 0$$

which is linear, integrating fact $\mu = (1+x)^2 \Rightarrow$

$$y_1(x) = \frac{c_1}{(1+x)^2} + \frac{1}{3}(1+x)$$

Initial Cond:

$$y_1(x) = \frac{1}{3} \left\{ (1+x) - \frac{1}{(1+x)^2} \right\}$$