Math 450 (2009) – Final (Take home)

Due: December 16, 2009 (10am) NAME: ________________________

Get the exam no me no later than 10am on December 16, 2009. You may give me the exam in person at my office (Wil 2-136), slide it under my office door or put it in my mailbox in the main math office.

---

1. [25 pts] Let $y(t, \epsilon)$ be the solution of the initial value problem

$$\begin{aligned} y'' + y &= \epsilon y^5 \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 0 \quad , \quad y'(0) = 1 + 3\epsilon \end{aligned}$$

where $(\ )'$ denotes differentiation in t. Assume

$$\begin{aligned} y(t, \epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon)t \equiv (1 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \cdots)t \end{aligned}$$

a) Use Poincare-Lindstedt's method to determine ω_1 and the $O(\epsilon)$ correction to the period of the oscillation. You may use the identity:

$$\sin^5 A = \frac{5}{8} \sin A - \frac{5}{16} \sin 3A + \frac{1}{16} \sin 5A$$

b) What initial conditions must $y_1(\tau)$ satisfy? Do not compute $y_1(\tau)$.

2. [25 pts] The following equation has two roots for positive ϵ.

$$\epsilon x^4 + \frac{1}{\sqrt{x}} = x \quad , \quad 0 < \epsilon \ll 1$$

Find a two term expansion in ϵ for the singular root $x = \bar{x}(\epsilon)$. Make sure you balance the largest two terms. Also, you may use the binomial expansion:

$$(X_0 + \delta X_1 + \cdots)^p = X_0^p + p\, X_0^{p-1} X_1\, \delta + O(\delta^2) \quad , \quad \delta \ll 1$$

3. [25pts] Let $y(x, \epsilon)$ be the solution of the following boundary value problem:

$$\begin{aligned} \epsilon y'' + (x+2)y' + y^2 &= 0 \quad , \quad x \in (0,1) \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= A \quad , \quad y(1) = \frac{1}{ln(3)} \end{aligned}$$

a) Find a uniformly valid approximation $y_u(x, \epsilon)$ of the solution for arbitrary A.

b) For what value of A is there no boundary layer at $x = 0$?

4. [25 pts] A functional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$\begin{aligned} J(y) &= \int_0^1 L(x, y(x), y'(x)) \, dx \\ \mathcal{A} &= \{y \in C^2[0,1] : y(0) = 2, y(1) = 3\} \end{aligned}$$

where the Lagrangian

$$L(x, y, y') = y \; y' \; ln(y')$$

Use a first integral of the Euler-Lagrange equations to find the extremal $\bar{y} \in \mathcal{A}$ of the functional J. You may assume $\bar{y}$ and $\bar{y}'$ are not negative.