Math 450 (2009) – Final (Take home)

Due: December 16, 2009 (10am) NAME: ____________________

Get the exam no me no later than 10am on December 16, 2009. You may give me the exam in person at my office (Wil 2-136), slide it under my office door or put it in my mailbox in the main math office.

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1. [25 pts] Let $y(t,\epsilon)$ be the solution of the initial value problem

$$\begin{aligned} y'' + y &= \epsilon y^5 \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 0 \quad , \quad y'(0) = 1 + 3\epsilon \end{aligned}$$

where $(\ )'$ denotes differentiation in t. Assume

$$\begin{aligned} y(t,\epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon)t \equiv (1 + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots)t \end{aligned}$$

a) Use Poincare-Lindstedt's method to determine ω_1 and the $O(\epsilon)$ correction to the period.

You may use the identity:

$$\sin^5 A = \frac{5}{8}\sin A - \frac{5}{16}\sin 3A + \frac{1}{16}\sin 5A$$

b) What initial conditions must $y_1(\tau)$ satisfy?

2. [25 pts] The following equation has two roots for positive ϵ.

$$\epsilon x^4 + \frac{1}{\sqrt{x}} = x \quad , \quad 0 < \epsilon \ll 1$$

Find a two term expansion in ϵ for the singular root. Make sure you balance the largest two terms. Also, you may use (for $p = 4, -\frac{1}{2}$):

$$(X_0 + \delta X_1 + \cdots)^p = X_0^p + pX_0^{p-1}X_1\,\delta + O(\delta^2)$$

3. [25pts] Let $y(x, \epsilon)$ be the solution of the following boundary value problem:

$$\begin{aligned} \epsilon y'' + (x+2)y' + y^2 &= 0 \quad , \quad x \in (0,1) \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= A \quad , \quad y(1) = \frac{1}{ln(3)} \end{aligned}$$

a) Find a uniformly valid approximation for arbitrary A (layer is at $x = 0$). Here the outer problem is separable and the right boundary condition looks odd to make $y_0(x)$ nice.

b) For what value of A is there no boundary layer at $x = 0$?

4. [25 pts] A functional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$\begin{aligned} J(y) &= \int_0^1 L(x, y(x), y'(x))\, dx \\ \mathcal{A} &= \{y \in C^2[0,1] : y(0) = 0, y(1) = 1\} \end{aligned}$$

where the Lagrangian

$$L(x, y, y') = y\ y'\ ln(y')$$

Use a first integral of the Euler-Lagrange equations to find the extremal $\bar{y} \in \mathcal{A}$ of the functional J. The first integral should be a separable first order differential equation for $\bar{y}(x)$. The solution will depend on two constants which you will determine from the boundary conditions defining $\mathcal{A}$. You may assume $\bar{y}$ and $\bar{y}'$ are both positive.

QUESTION ONE

Differential eqns for $y_0(\tau)$ and $y_1(\tau)$ are:

$$(1) \qquad y_0'' + y_0 = 0$$

$$(2) \qquad y_1'' + y_1 = y_0^5 - 2\omega_1 y_0''$$

Expand initial conditions

$$y(0,\varepsilon) = 0 = y_0(0) + \varepsilon y_1(0) + O(\varepsilon^2)$$

$$y'(0,\varepsilon) = 1 + 3\varepsilon = y_0'(0) + \varepsilon(y_1'(0) + \omega_1 y_0'(0)) + O(\varepsilon^2)$$

Comparing powers of ε

$$(3) \qquad y_0(0) = 0 \qquad y_0'(0) = 1$$

$$(4) \qquad y_1(0) = 0 \qquad y_1'(0) = 3 - \omega_1 y_0'(0)$$

Solution of $O(1)$ problem (1), (3) is

$$y_0(\tau) = \sin \tau$$

so that (2) becomes (using the identity)

$$y_1'' + y_1 = \overset{0}{\left(\frac{5}{8} - 2\omega_1\right)} \sin \tau - \frac{5}{16} \sin 3\tau + \frac{1}{16} \sin 5\tau$$

Thus

$$(5) \qquad \omega_1 = \frac{5}{16} \qquad T = \frac{2\pi}{\omega} = 2\pi\left(1 - \frac{5}{16}\varepsilon + O(\varepsilon^2)\right)$$

Using ω_1 in (5) in (4)

$$y_1'(0) = 3 - \frac{5}{16}(1) = \frac{43}{16}$$

QUESTION TWO $\qquad x = \dfrac{X}{\varepsilon^{\alpha}}$, $\alpha > 0$

$$\varepsilon^{1-4\alpha} X^4 + \varepsilon^{\alpha/2} X^{-1/2} = \varepsilon^{-\alpha} X$$

$$\quad (1) \qquad\qquad (2) \qquad\qquad (3)$$

For all $\alpha > 0$ term (2) $\ll$ (3) hence need (1) $\sim$ (3)

$$1 - 4\alpha = -\alpha$$

$$\alpha = \tfrac{1}{3}$$

For this choice

$$X^4 - X = -\delta(\varepsilon) X^{-1/2}$$

where $\delta(\varepsilon) = \varepsilon^{1/2}$. Expansion

$$X = X_0 + \delta X_1 + O(\delta^2) \qquad \delta = \varepsilon^{1/2} \ll 1$$

yields

$$X_0^4 - X_0 = 0 \qquad \Rightarrow X_0 = \cancel{0}, 1$$

$$4X_0^3 X_1 - X_1 = -X_0^{-1/2}$$

For $X_0 = 1$, $X_1 = -\tfrac{1}{3}$ and

$$X = 1 - \tfrac{1}{3}\varepsilon^{1/2} + O(\varepsilon)$$

or,

$$x = \varepsilon^{-1/3}\left(1 - \tfrac{1}{3}\varepsilon^{1/2} + O(\varepsilon)\right)$$

QUESTION THREE

Outer problem

$$(x+2)\,y_0' + y_0^2 = 0 \qquad y_0(1) = \frac{1}{\ln 3}$$

Since the general solution is

$$y_0(x) = (c_1 + \ln(x+2))^{-1}$$

we require $c_1 = 0$ for the boundary condition to be satisfied:

$$y_0(x) = \frac{1}{\ln(x+2)}$$

Inner Problem

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\delta(\varepsilon)}$$

yields

$$\frac{\varepsilon}{\delta^2} \underset{\uparrow}{Y''} + \underset{\uparrow}{(2+\delta X)}\cdot\frac{1}{\delta} Y' + Y^2 = 0$$

Balancing indicated (largest) terms $\delta(\varepsilon) = \varepsilon \Rightarrow$

$$Y'' + (2+\varepsilon X)\,Y' + \varepsilon Y^2 = 0$$

The expansion $Y(X,\varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \cdots \Rightarrow$

$$Y_0'' + 2Y_0' = 0$$

$$Y_0(0) = A$$

whose solution is

$$Y_0(X) = A + C_2 (e^{-2X} - 1)$$

for some constant C_2 to be determined by matching

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

$$M = \frac{1}{\ln 2} = A - C_2$$

From which $C_2 = A - (\ln 2)^{-1}$.

$$Y_0(X) = A + \left(A - \frac{1}{\ln 2}\right)(e^{-2X} - 1)$$

Uniform approximation

$$y_u(x, \varepsilon) = y_0(x) + Y_0(x/\varepsilon) - M$$

$$y_u(x, \varepsilon) = \frac{1}{\ln(x+2)} + Y_0(x/\varepsilon) - \frac{1}{\ln 2}$$

Part b)

There is no layer if the outer solution satisfies both boundary conditions, i.e. if A is chosen so

$$y_0(0) = \frac{1}{\ln 2} = A$$

QUESTION FOUR

Since $L_x = 0$ a first integral is

$$L - y' L_{y'} = -k_1$$

for some constant k_1. For $L = yy' \ln(y')$

$$yy' \cancel{\ln} y' - y'(y \cancel{\ln} y' + y) = -k_1$$

Hence

$$yy' = k_1$$

which is separable and has soln

$$y(x) = \sqrt{Ax + B}$$

Here we have assumed $y(x) \geq 0$. For $y \in A$ it needs to satisfy the boundary conditions

$$y(0) = \sqrt{B} = 2$$

$$y(1) = \sqrt{A + B} = 3$$

From which $B = 4$, $A = 5$ and

$$\bar{y}(x) = \sqrt{5x + 4}$$