

2011
Math 450 (~~2009~~) – Homework 1

Due: September 16, 2011.

NAME: _____

1. [20pts] Find the solution of the following initial value problems:

$$y' - \frac{1}{2x} y = \sqrt{x} \quad , \quad y(1) = 4 \quad (1)$$

$$y' + \frac{1}{x} y = \frac{5x}{y^2} \quad , \quad y(1) = 2 \quad (2)$$

$$y'' - 2y' + 5y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad (3)$$

$$y'' + 2y' + y = 1 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 2 \quad (4)$$

2. [5pts] Use the method of Variation of Parameters to find a particular solution of

$$y'' - 2y' + y = x^3 e^x$$

3. [5pts] Let $f(x, y) = y - x^3$. Find that unique curve through $(x, y) = (1, 1)$ that is orthogonal to the level curves of f . Sketch several $f = c$ level curves and the resulting orthogonal curve (just the first quadrant).

4. [10pts] Find a Fundamental Matrix $X(t) \in \mathbb{R}^{2 \times 2}$ of the system

$$\frac{dx}{dt} = Ax$$

for the following two matrices:

$$A = \begin{bmatrix} -7 & -6 \\ 4 & 7 \end{bmatrix} \quad , \quad A = \begin{bmatrix} -8 & 4 \\ -10 & 4 \end{bmatrix}$$

QUESTION ONE a) (Linear)

$$y' - \frac{1}{2x} y = \sqrt{x} \quad y(1) = 4$$

Integrating factor

$$\mu(x) = \exp\left(-\int \frac{1}{2t} dt\right) = \frac{1}{\sqrt{x}}$$

Hence

$$y(x) = \frac{c}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(t) q(t) dt$$

$$y(x) = c\sqrt{x} + x^{3/2}$$

is the general soln. $y(1) = c + 1 = 4 \Rightarrow c = 3$

$$y(x) = 3\sqrt{x} + x^{3/2}$$

QUESTION ONE b) (Bernoulli)

$$y' + \frac{1}{x} y = 5x y^{-2}$$

$$(1) \quad y' + p(x)y = q(x)y^n \quad n = -2$$

Use

$$u = y^{1-n} = y^3$$

to get linear eqn for $u(x)$

$$u' + \frac{3}{x} u = 15x$$

whose general soln is $u(x) = 3x^2 + \frac{c}{x^3} = y^3$
Since $y(1) = 2$, $u(1) = 8$ and $c = 5$

$$y(x) = \sqrt[3]{3x^2 + \frac{5}{x^3}} = \frac{1}{x} \sqrt[3]{3x^5 + 5}$$

QUESTION ONE (c) (Complex roots)

$$y'' - 2y' + 5y = 0 \quad y(0) = 0, y'(0) = 1$$

has characteristic polynomial $P(\lambda) = \lambda^2 - 2\lambda + 5$.
Its roots are $\lambda = 1 \pm 2i$ so that the
general solution is

$$y(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$$

Since $y(0) = c_1 = 0$ we have $y(x) = c_2 e^x \sin(2x)$
and easy to show $y'(0) = 2c_2 = 1$ so unique
solution

$$y(x) = \frac{1}{2} e^x \sin(2x)$$

QUESTION ONE (d)

$$y'' + 2y' + y = 1 \quad y(0) = 1, y'(0) = 2$$

Char. Poly $P(\lambda) = (\lambda + 1)^2 \Rightarrow \lambda = -1$ repeated.
Particular soln $y_p(x) = 1$. General Soln

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + 1$$

$$y'(x) = -c_1 e^{-x} + c_2 (1-x) e^{-x}$$

Apply initial conditions

$$\left. \begin{aligned} y(0) &= c_1 + 1 &= 1 \\ y'(0) &= -c_1 + c_2 &= 2 \end{aligned} \right\} \begin{aligned} c_1 &= 0 \\ c_2 &= 2 \end{aligned}$$

unique soln

$$y(x) = 2x e^{-x} + 1$$

QUESTION TWO

$$y'' - 2y' + y = x^3 e^x = f(x)$$

Homogenous solutions for $P(\lambda) = (\lambda - 1)^2$ are

$$y_1(x) = e^x \quad y_2(x) = x e^x$$

Wronskian

$$W = y_1 y_2' - y_2 y_1' = e^{2x}$$

Particular solution

$$(1) \quad y_p(x) = a(x) y_1(x) + b(x) y_2(x)$$

where

$$a(x) = - \int \frac{1}{W(t)} y_2(t) f(t) dt = -\frac{1}{5} x^5$$

$$b(x) = + \int \frac{1}{W(t)} y_1(t) f(t) dt = +\frac{1}{4} x^4$$

using this in (1) we get

$$y_p(x) = \frac{1}{20} x^5 e^x$$

QUESTION THREE (Level curves)

$$(1) \quad f(x, y) = y - x^3 = c$$

are cubics $y = x^3 + c$. Orthogonal level curves satisfy ODE

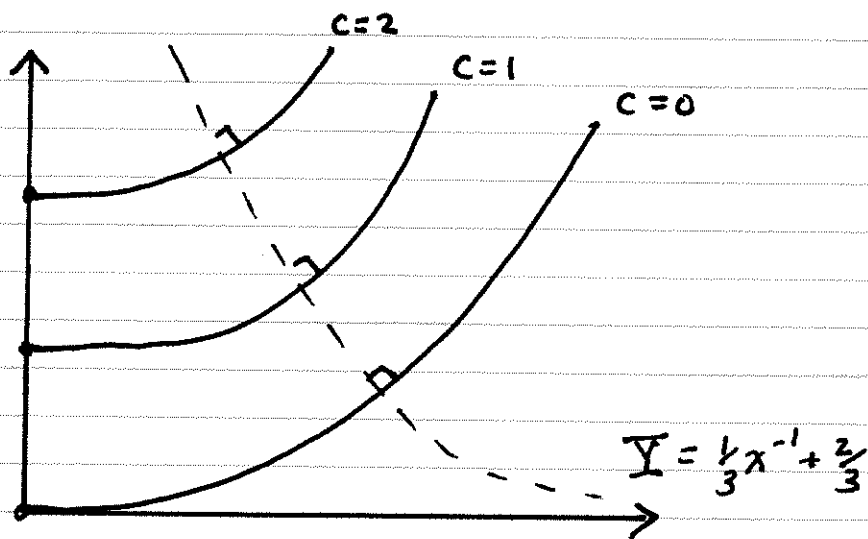
$$(2) \quad \frac{dy}{dx} = \frac{f_y}{f_x} = -\frac{1}{3x^2}$$

whose general soln is

$$y(x) = +\frac{1}{3}x^{-1} + k$$

The one thru $(x, y) = (1, 1)$ has $y(1) = 1 \Rightarrow k = \frac{2}{3}$

$$y(x) = \frac{1}{3}x^{-1} + \frac{2}{3}$$



QUESTION FOUR (a)

$$A = \begin{bmatrix} -7 & -6 \\ 4 & 7 \end{bmatrix}$$

Char. Poly $P(\lambda) = \det(A - \lambda I) = \lambda^2 - 25.$

Eigenvalues $\lambda_1 = 5$ $\lambda_2 = -5$ (real distinct)

$$A - \lambda_1 I = \begin{bmatrix} -12 & -6 \\ 4 & 2 \end{bmatrix} \quad \vec{z}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{evector}$$

$$A - \lambda_2 I = \begin{bmatrix} -2 & -6 \\ 4 & 12 \end{bmatrix} \quad \vec{z}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{evector}$$

Fundamental Matrix

$$X(t) = \begin{bmatrix} e^{\lambda_1 t} \vec{z}_1 & e^{\lambda_2 t} \vec{z}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & -3e^{-5t} \\ -2e^{5t} & e^{-5t} \end{bmatrix}$$

QUESTION FOUR (b)

$$A = \begin{bmatrix} -8 & 4 \\ -10 & 4 \end{bmatrix} \quad P = \lambda^2 + 4\lambda + 8$$

Eigenvalue(s) complex $\lambda = -2 + 2i$ (root of P)

$$(A - \lambda I) \vec{z} = \begin{bmatrix} -6 - 2i & 4 \\ -10 & 6 - 2i \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \vec{0}$$

Nonunique \vec{z} . Pick $z_1 = 1 \Rightarrow z_2 = \frac{3}{2} + \frac{1}{2}i$

$$\lambda = -2 + 2i \quad \vec{z} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} i = \vec{a} + i\vec{b}$$

Fundamental Matrix

$$X(t) = \begin{bmatrix} e^{-2t} \cos(2t) & e^{-2t} \sin(2t) \\ e^{-2t} \left(\frac{3}{2} \cos(2t) - \sin(2t) \right) & e^{-2t} \left(\frac{3}{2} \sin(2t) + \cos(2t) \right) \end{bmatrix}$$