

Math 450 (2009) – Homework 1

Due: September 18, 2009.

NAME: _____

1. [20pts] Find the solution of the following initial value problems:

$$y' + \tan(x) y = \sec(x) \quad , \quad y(0) = 1 \quad (1)$$

$$xy' + y = xy^3 \quad , \quad y(1) = 1 \quad (2)$$

$$y'' + 2y' + y = 0 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 1 \quad (3)$$

$$y'' - 4y' + 13y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad (4)$$

2. [5pts] Use the method of Variation of Parameters to find a particular solution of

$$\frac{d^2y}{dx^2} + y = \tan(x)$$

3. [5pts] Level curves of a function $f(x, y)$ have tangents whose slope m is given by implicit differentiation:

$$f(x, y) = c \quad \Rightarrow \quad f_x + f_y \frac{dy}{dx} = 0 \quad \Rightarrow \quad m = \frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$$

Thus, curves that are everywhere orthogonal to level curves have a slope that satisfies the differential equation

$$\frac{dy}{dx} = -\frac{1}{m} = \frac{f_y(x, y)}{f_x(x, y)}$$

Use this theory to find the equation of the curves orthogonal to the curves $x^4 + y^4 = c$.

Sketch the level curves for a few values of $c > 0$ as well as a few orthogonal curves.

4. [5pts] For $a, b > 0$ the logistic equation

$$\frac{dP}{dt} = aP - bP^2 = \text{birth rate} - \text{death rate}$$

well models some organism populations $P(t)$ at time t . Find a formula for $P(t)$ given the initial population $P(0) = P_0$.

Homework 1

Question one (a) linear 1st order

$$y' + \tan(x)y = \sec x \quad y(0) = 1$$

Integrating factor

$$\mu(x) = \exp\left(\int \tan t \, dt\right) = \sec x$$

Particular solution

$$y_p(x) = \frac{1}{\mu(x)} \int \mu(t) f(t) \, dt = \cos x \int \sec^2 t \, dt$$

$$y_p(x) = \cos x \tan x = \sin x$$

General Solution ($y_h(x) = \mu(x)^{-1}$)

$$y(x) = c \cos x + \sin x$$

Initial Conditions

$$y(0) = c \cos 0 + \sin 0 = 1$$

implies $c = 1$ so

$$y(x) = \cos x + \sin x$$

Question one (b) Bernoulli equation.

$$y' + \frac{1}{x}y = y^3 \quad y(1) = 1$$

Let $u = y^{-2}$ and divide by y^3 to get

$$-\frac{1}{2}u' + \frac{1}{x}u = 1$$

$$u' - \frac{2}{x}u = -2$$

Linear equation in u with integrating factor

$$\mu(x) = \exp\left(-\int \frac{2}{t} dt\right) = \frac{1}{x^2}$$

Particular solution

$$u_p(x) = \frac{1}{\mu(x)} \int \mu(t) f(t) dt = x^2 \int \left(\frac{-2}{t^2}\right) dt = x^2 \left(\frac{2}{x}\right) = 2x$$

Thus general solution for $u(x)$

$$u(x) = c \cdot \frac{1}{\mu(x)} + u_p(x)$$

$$u(x) = 2x + cx^2$$

General solution for $y(x)$

$$y(x) = \frac{1}{\sqrt{u(x)}} = \frac{1}{\sqrt{2x + cx^2}}$$

Initial conditions

$$y(1) = \frac{1}{\sqrt{2+c}} = 1 \quad c = -1$$

implies

$$y(x) = \frac{1}{\sqrt{2x - x^2}}$$

Question one (c)

2nd order linear.

$$y'' + 2y' + y = 0$$

$$y(0) = 2 \quad y'(0) = 1$$

Characteristic Polynomial

$$P(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

Repeated root $\lambda_1 = \lambda_2 = -1$. General Soln

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

from which

$$y'(x) = -c_1 e^{-x} + c_2 (e^{-x} - x e^{-x})$$

Using initial conditions

$$y(0) = c_1 = 2$$

$$y'(0) = -c_1 + c_2 = 1$$

Hence $c_1 = 2$ and $c_2 = 3 \Rightarrow$

$$y(x) = 2e^{-x} + 3xe^{-x}$$

solves the initial value problem.

Question one (d)

2nd order linear

$$y'' - 4y' + 13y = 0$$

$$y(0) = 0 \quad y'(0) = 1$$

Characteristic Polynomial

$$P(\lambda) = \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{1}{2}(4 \pm \sqrt{16 - 52}) = \frac{1}{2}(4 \pm \sqrt{-36})$$

Thus complex $\lambda = \alpha \pm i\beta = 2 \pm 3i$. General Soln,

$$y(x) = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$$

From which

$$y'(x) = C_1 (2e^{2x} \cos 3x - 3e^{2x} \sin 3x)$$

$$+ C_2 (2e^{2x} \sin 3x + 3e^{2x} \cos 3x)$$

Using initial conditions

$$y(0) = C_1 = 0$$

$$y'(0) = 2C_1 + 3C_2 = 1$$

Hence $C_1 = 0$, $C_2 = \frac{1}{3}$ and

$$y(x) = \frac{1}{3} e^{2x} \sin(3x)$$

solves IVP.

QUESTION TWO

Variation of Parameters

$$y'' + y = \tan x$$

Characteristic Polynomial $P(\lambda) = \lambda^2 + 1$ has imaginary roots $\lambda = \pm i$ hence homog. solns are:

$$y_1(x) = \cos x \quad y_2(x) = \sin x$$

$$\text{Wronskian } W = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1.$$

Particular Solution

$$y_p(x) = a(x)y_1(x) + b(x)y_2(x)$$

where

$$a(x) = - \int \frac{y_2(t)f(t)}{W(t)} dt = - \int \sin t \tan t dt$$

$$b(x) = + \int \frac{y_1(t)f(t)}{W(t)} dt = + \int \cos t \tan t dt$$

Clearly $b(x) = -\cos x$. The integral for $a(x)$ is harder. Using $\sin^2 t = 1 - \cos^2 t$, $\tan t = \sin t / \cos t$ and tables

$$a(x) = - \int (\csc t - \cos t) dt = \sin x - \ln(\sec x + \tan x)$$

From which we conclude

$$y_p(x) = \sin x / \cos x - \cos x \ln(\sec x + \tan x) - \sin x / \cos x$$

$$y_p(x) = -\cos x \ln(\sec x + \tan x)$$

QUESTION THREE

Orthogonal Curves

Given the family of curves

$$f(x, y) = x^4 + y^4 = c \quad > 0$$

curves that are everywhere orthogonal must satisfy

$$\frac{dy}{dx} = \frac{f_y(x, y)}{f_x(x, y)} = \frac{y^3}{x^3}$$

which is a separable 1st order diff eqn.

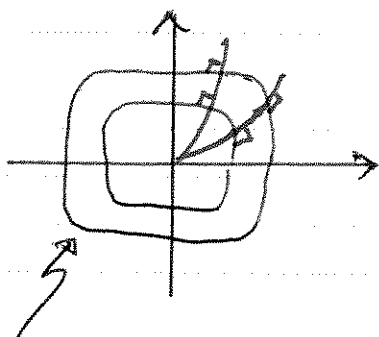
$$\int y^{-3} dy = \int x^{-3} dx$$

$$-\frac{1}{2y^2} = -\frac{1}{2x^2} + k$$

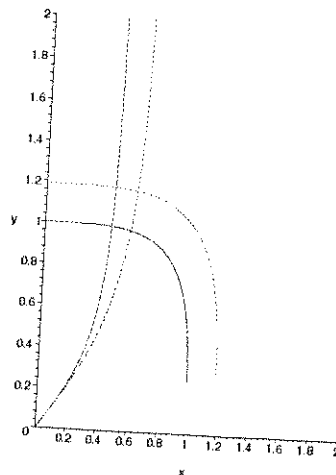
for constant k . Solve for $y(x)$

$$y(x) = \pm \frac{x}{\sqrt{1 - 2kx^2}}$$

Plots



level curves look like rounded squares



QUESTION FOUR

Logistic Equation

$$P'(t) = aP - bP^2 \quad P(0) = P_0$$

where $P(t)$ = population. Separable eqn:

$$\int_{P_0}^P \frac{dz}{az - bz^2} = \int_0^t ds$$

Partial Fraction expansion of left side

$$\int_{P_0}^P \left(\frac{1}{az} - \frac{b}{a(bz-a)} \right) dz = t$$

leads to

$$\ln(H(z)) \Big|_{P_0}^P = \ln\left(\frac{bz-a}{z}\right) \Big|_{P_0}^P = -at$$

Algebraically

$$H(P)/H(P_0) = e^{-at}$$

which one solves for P .

$$P(t) = \frac{a}{b + \left(\frac{a}{P_0} - b\right)e^{-at}}$$

Note $P(t) \rightarrow \frac{a}{b}$ as $t \rightarrow \infty$.