Math 450 (2011) – Homework 2

Due: October 7, 2011. NAME: ____________________

1. [5pts] This problem considers the period p of a pendulum. A mass m attached to a string of length ℓ swings under the influence of gravity. The gravitational constant is g. The maximum angle the string makes with the vertical axis is θ. Although angles (in radians) are dimensionless, we shall retain it as a dimensionless quantity $\Pi_1 = \theta$. We shall assume

$$f(p, m, \ell, g, \theta) = 0$$

for some function f.

a) Use dimensional analysis so that $f = 0$ is equivalent to

$$F(\Pi_1, \Pi_2) = 0$$

for some dimensionless Π_2 and function F. In particular find Π_2 and conclude

$$\Pi_2 = \phi(\theta)$$

for a function ϕ to be determined from experiment.

b) The surface gravitational constants for the earth and moon are $g_e = 9.8 m/sec^2$ and $g_m = 1.62 m/sec^2$, respectively. If the period of the pendulum on earth is one second, what will it be on the moon? Use you result in part a) to answer this. Do not consult a physics book.

2. [10pts] The following problem involves the impact an deformation of an elastic ball as it hits a wall. Suppose a ball of diameter D, and density ρ is travelling velocity v just before it strikes a wall. The elastic properties of the ball is determined by the modulus of elasticity E where $[E] = [\rho v^2]$. Use dimensional analysis to determine the deformed diameter d as a function of other quantities assuming the law

$$f(D, d, \rho, V, E) = 0$$

for some function f. Your answer should look something like

$$d = stuff \times (some\ function\ of\ dimensionless\ quantities)$$

3. [10pts] Boltzman studied the electromagetic energy radiated by substances at different temperatures τ. He believed such radiated energy could only be explained with both quantum mechanics and thermodynamics so should involve the key parameters: Planck's constant $\hbar$ and Boltzmann's constant k where $[k] = joules/K$ where $K =^\circ Kelvin$. Planck's constant has units of joules $\times$ time and the radiation travels at the speed of light c. The energy <u>density</u> $\mathcal{E}$ $(joules/L^3)$ he thought was related to the former parameters. Assume

$$f(\mathcal{E}, \tau, \hbar, k, c) = 0 \quad ,$$

for some function f and then use dimensional analysis to find a formula for $\mathcal{E}$ in terms of dimensional quantities. What power of τ is $\mathcal{E}$ proportional to?

4. [10pts] The population of bacteria in a container of volume V are a fed soluble nutrient at a constant rate F (volume per time). A set of differential equations which models the bacteria concentration $N(t)$ and nutrient concentration $C(t)$ is:

$$\begin{aligned} \frac{dN}{dt} &= \left(\frac{K_{max}C}{K_n + C}\right) N - \frac{FN}{V} \\ \frac{dC}{dt} &= -\alpha \left(\frac{K_{max}C}{K_n + C}\right) N - \frac{FC}{V} + \frac{FC_0}{V} \end{aligned}$$

Here $[N] = ML^{-3}, [C] = ML^{-3}$ and the parameters $K_{max}, K_n, F, V, \alpha, C_0$ are constant. We won't worry about why these equations are appropriate but seek to simplify their analysis by first nondimensionalizing them.

a) Determine the dimensions of $K_{max}, K_n, F, V, \alpha, C_0$ in terms of the fundamental units M, L, T.

b) Define dimensionless variables

$$n = \frac{N}{N^*} \quad , \quad c = \frac{C}{C^*} \quad , \quad \tau = \frac{t}{t^*}$$

Find N^*, C^* and t^* such that the nondimensionalized system is

$$\begin{aligned} \frac{dn}{d\tau} &= \alpha_1 \left(\frac{c}{1+c}\right) n - n \\ \frac{dc}{d\tau} &= -\left(\frac{c}{1+c}\right) n - c + \alpha_2 \end{aligned}$$

Write out formulae defining α_1, α_2. Note that the original system had 6 parameters versus 2 for the dimensionless system!

5. [5pts] Consider a physical law involving four quantities:

$$f(q_1, q_2, q_3, q_4) = 0$$

where each is expressible in terms of three fundamental units L_1, L_2, L_3. The quantity

$$\Pi = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3} q_4^{\alpha_4}$$

is dimensionless only if $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is in the nullspace of a matrix $A \in \mathbb{R}^{3\times 4}$. We'll suppose that $dim N(A) = 2$ and that you've found two basis vectors $\vec{a}$ and $\vec{b}$ so that

$$\begin{aligned} \Pi_a &= q_1^{a_1} q_2^{a_2} q_3^{a_3} q_4^{a_4} \\ \Pi_b &= q_1^{b_1} q_2^{b_2} q_3^{b_3} q_4^{b_4} \end{aligned}$$

are dimensionless. Your friend finds a different basis vector:

$$\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b}$$

and associated dimensionless quantity Π_c. Show

$$\Pi_c = \Pi_a^{\lambda_1} \Pi_b^{\lambda_2}$$