

## Math 450 (2009) – Homework 2

Due: October 2, 2009.

NAME: \_\_\_\_\_

1. [6pts] Deep water oceanic waves of wavelength  $\lambda$  are observed to travel at a constant speed  $v$ . Because these waves are deep they are thought to be primarily a gravitational phenomena. Assume that  $\lambda$ ,  $v$  and the gravitational constant  $g$  are related:

$$f(\lambda, v, g) = 0$$

a) Find all the dimensionless parameters  $\pi$  associated with the problem where

$$\pi = \lambda^{\alpha_1} v^{\alpha_2} g^{\alpha_3}$$

b) Use the  $\pi$ -theorem to derive a formula for  $v$  in terms of the other dimensional quantities. What power of  $\lambda$  is  $v$  proportional to?

2. [12pts] Bubbles rise in fluids. If you've ever seen videos of scuba divers you should have noticed large bubbles rise faster. Thus, the bubble speed  $v$  depends on the bubble volume  $V$ . The density  $\rho_0$  of the gas in the bubble, the fluid density  $\rho$  and gravity  $g$  all affect the bubble velocity. Assume all the aforementioned dimensional quantities are related:

$$f(\rho, \rho_0, V, v, g) = 0$$

a) Find all the dimensionless parameters  $\pi$  associated with the problem where

$$\pi = \rho^{\alpha_1} \rho_0^{\alpha_2} V^{\alpha_3} v^{\alpha_4} g^{\alpha_5}$$

b) Use the  $\pi$ -theorem to derive a formula for  $v$  in terms of the other dimensional quantities.

c) If a bubble of volume  $V$  rises at  $4\text{cm}/\text{sec}$  then how fast does a bubble of four times the volume rise?

3. [4pts] Let  $q_1, q_2$  be two dimensional quantities and  $[q_1] = L_1^a L_2^b$ ,  $[q_2] = L_1^c L_2^d$  ( $a, b, c, d > 0$ ) in terms of some fundamental dimensions  $L_1, L_2$ . Show that there are nontrivial dimensionless  $\pi = q_1^{\alpha_1} q_2^{\alpha_2}$  if and only if  $\frac{a}{b} = \frac{c}{d}$  and that any resulting physical law can be written  $q_1 = k q_2^{\frac{a}{b}}$  for some constant  $k$ .

4. [8pts] Consider an ecosystem of herbivores (plant eaters) and plants. Let  $H$  and  $P$  be the total carbon biomass (kg) of the herbivores and plants, respectively. Plants produce carbon through photosynthesis at a rate  $\phi$ . A dimensional model of the herbivore-plant carbon mass dynamic is

$$\frac{dP}{d\tau} = \phi - aP - bHP \quad (1)$$

$$\frac{dH}{d\tau} = \epsilon bHP - cH \quad (2)$$

where  $\tau$  is dimensional time and  $a, b, c, \epsilon, \phi$  are all positive constants. The term  $\epsilon bHP$  represents the rate at which the herbivores gain carbon biomass by eating the plants. Likewise, the term  $bHP$  represents the loss of biomass in the the total plant population due to the herbivores eating them. The term  $cH$  represents the rate at which the herbivores secrete biomass. Plants also lose biomass due to death and other mechanisms as reflected in the term  $aP$ .

- a) Determine the units of  $a, b, c, \epsilon$  and  $\phi$  in terms of the fundamental dimensions mass  $M$  and time  $T$ .
- b) Nondimensionalize the model using the lowercase naming convention:

$t$  = dimensionless time

$p$  = dimensionless plant biomass

$h$  = dimensionless herbivore biomass

Choose your scaling so the equation for  $\frac{dp}{dt}$  has no parameters whatsoever.

### QUESTION 1

Deep water waves of wavelength  $\lambda$  travel at velocity  $v$  and are predominantly a gravitational phenomena.

Assume wavelength  $\lambda$ , velocity  $v$  and gravitational constant related.

$$(1) \quad f(\lambda, v, g) = 0$$

Units

$$[\lambda] = L \quad [v] = L T^{-1} \quad [g] = L T^{-2}$$

Dimensionless parameters

$$\pi = \lambda^{\alpha_1} v^{\alpha_2} g^{\alpha_3}$$
$$[\pi] = L^{\alpha_1 + \alpha_2 + \alpha_3} T^{-\alpha_2 - 2\alpha_3}$$

Thus

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$
$$\alpha_2 + 2\alpha_3 = 0$$

One free parameter. Set  $\alpha_3 = 1$

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (1, -2, 1)$$

$$\pi = \frac{\lambda g}{v^2} \quad \text{sole dimensionless param.}$$

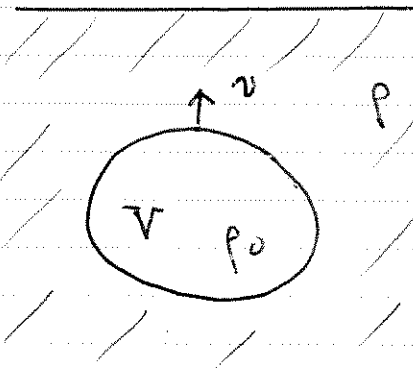
$\pi$ -Theorem  $\Rightarrow$  Law (1) is equivalent to  $F(\pi) = 0$  for some function  $F$ . Since  $\pi$  is a root we conclude

$$v = c \sqrt{\lambda g}$$

for some (experimentally measurable) constant  $c$ .

## QUESTION 2

## Bubble Problem



Gas bubble floats up in a fluid due to buoyancy.

$\rho$  = fluid density

$\rho_0$  = gas density

$V$  = bubble volume

$v$  = terminal velocity

Problem should depend on gravity  $g$ .

Assume a law of the form

$$(1) \quad f(\rho, \rho_0, V, v, g) = 0$$

STEP ONE Find dimensionless quantities.

$$[\rho] = ML^{-3}$$

$$[\rho_0] = ML^{-3}$$

$$[V] = L^3$$

$$[v] = LT^{-1}$$

$$[g] = LT^{-2}$$

Set dimensionless  $\pi$  equal to

$$\pi = \rho^{\alpha_1} \rho_0^{\alpha_2} V^{\alpha_3} v^{\alpha_4} g^{\alpha_5}$$

$$[\Pi] = L^{-3\alpha_1 - 3\alpha_2 + 3\alpha_3 + \alpha_4 + \alpha_5} M^{\alpha_1 + \alpha_2} T^{-\alpha_4 - 2\alpha_5}$$

Thus  $\Pi$  dimensionless iff

$$(2) \quad 3\alpha_1 + 3\alpha_2 - 3\alpha_3 - \alpha_4 - \alpha_5 = 0$$

$$(3) \quad \alpha_1 + \alpha_2 = 0$$

$$(4) \quad \alpha_4 + 2\alpha_5 = 0$$

Again solving this amounts to finding the nullspace  $N(A)$  of the coefficient matrix. Circled are the "free" variables

Eqs (3) - (4) imply

$$(5) \quad \alpha_1 = -\alpha_2 \quad \alpha_4 = -2\alpha_5$$

which used in (2) yield

$$(6) \quad \alpha_3 = \frac{1}{3}\alpha_5$$

$$\underline{(\alpha_2, \alpha_5)} = (1, 0)$$

$$\vec{\alpha} = (-1, +1, 0, 0, 0)$$

$$\pi_1 = \frac{\rho_0}{\rho}$$

$$\underline{(\alpha_2, \alpha_5)} = (0, 1)$$

$$\vec{\alpha} = (0, 0, \frac{1}{3}, -2, 1)$$

$$\pi_2 = \frac{V^{-1/3} g}{v^2}$$

## STEP TWO Make conclusions

By the  $\pi$ -Theorem the assumed law

$$f(\rho, \rho_0, V, v, g) = 0$$

is equivalent to

$$F(\pi_1, \pi_2) = 0$$

for some fn  $F$ . Thus for some (unknown) function  $\phi$

$$\pi_2 = \frac{V^{1/3} g}{v^2} = \phi(\pi_1)$$

Solving for  $v$ , there is some function  $\Phi$  such that

$$v = \sqrt{g V^{1/3}} \Phi\left(\frac{\rho_0}{\rho}\right)$$

Part (c) Using the above we have

$$4 = \sqrt{g V^{1/3}} \Phi\left(\frac{\rho_0}{\rho}\right)$$

thus for 4 times the volume

$$v = \sqrt{g (4V)^{1/3}} \Phi\left(\frac{\rho_0}{\rho}\right)$$

$$v = 4^{1/6} \sqrt{g V^{1/3}} \Phi\left(\frac{\rho_0}{\rho}\right)$$

$$v = 4^{1/6} \cdot 4 = 4^{7/6} \approx 5 \text{ cm/sec}$$

QUESTION 3 Given  $q_1, q_2$  with

$$[q_1] = L_1^a L_2^b \quad [q_2] = L_1^c L_2^d$$

Define  $\pi$  by

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2}$$

Then

$$[\pi] = L_1^{(a\alpha_1 + c\alpha_2)} L_2^{(b\alpha_1 + d\alpha_2)}$$

Thus  $\pi$  is dimensionless if and only if

$$(1) \quad A\vec{\alpha} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \vec{0}$$

Dimension matrix has nontrivial nullspace if and only if  $\det A = 0$

$$\det A = ad - bc = 0 \quad \Leftrightarrow \quad \frac{a}{b} = \frac{c}{d}$$

Part b) Many ways to show power law.  
Here's one

$\vec{\alpha} = (c, -a)$  solves (1) so the sole dimensionless  $\pi$  is constant:

$$\pi = q_1^c q_2^{-a} = \beta \quad (\text{constant})$$

Solving for  $q_1$  we find

$$q_1 = k q_2^\mu, \quad \mu = \frac{a}{c}$$

and  $k = \beta^{1/c}$ .

## QUESTION FOUR

(a)

$$\begin{aligned}[\phi] &= M T^{-1} \\ [a] &= T^{-1} \\ [b] &= M^{-1} T^{-1}\end{aligned}$$

$$\begin{aligned}[\varepsilon] &= 1 \text{ dimensionless} \\ [c] &= T^{-1}\end{aligned}$$

(b) Nondimensionalization

$$t = \frac{T}{T^*} \quad p = \frac{P}{P^*} \quad h = \frac{H}{H^*}$$

yields

$$\frac{dp}{dt} = \left( \frac{\phi T^*}{P^*} \right) - \left( a T^* \right) p - \left( b T^* H^* \right) p h$$

$$\frac{dh}{dt} = \varepsilon \left( b P^* T^* \right) h p - \left( c T^* \right) h$$

Pick  $T^*$ ,  $P^*$ ,  $H^*$  so terms ①-③ all equal one

$$T^* = \frac{1}{a} \quad P^* = \frac{\phi}{a} \quad H^* = \frac{a}{b}$$

resulting in

$$\begin{aligned}\frac{dp}{dt} &= 1 - p - p h \\ \frac{dh}{dt} &= \alpha h p - \beta h\end{aligned}$$

where

$$\alpha = \frac{\varepsilon b \phi}{a^2} \quad \beta = \frac{c}{a}$$