

Math 450 (2011) – Homework 2

Due: October 7, 2011.

NAME: _____

1. [5pts] This problem considers the period p of a pendulum. A mass m attached to a string of length ℓ swings under the influence of gravity. The gravitational constant is g . The maximum angle the string makes with the vertical axis is θ . Although angles (in radians) are dimensionless, we shall retain it as a dimensionless quantity $\Pi_1 = \theta$. We shall assume

$$f(p, m, \ell, g, \theta) = 0$$

for some function f .

a) Use dimensional analysis so that $f = 0$ is equivalent to

$$F(\Pi_1, \Pi_2) = 0$$

for some dimensionless Π_2 and function F . In particular find Π_2 and conclude

$$\Pi_2 = \phi(\theta)$$

for a function ϕ to be determined from experiment.

b) The surface gravitational constants for the earth and moon are $g_e = 9.8m/sec^2$ and $g_m = 1.62m/sec^2$, respectively. If the period of the pendulum on earth is one second, what will it be on the moon? Use you result in part a) to answer this. Do not consult a physics book.

2. [10pts] The following problem involves the impact an deformation of an elastic ball as it hits a wall. Suppose a ball of diameter D , and density ρ is travelling velocity v just before it strikes a wall. The elastic properties of the ball is determined by the modulus of elasticity E where $[E] = [\rho v^2]$. Use dimensional analysis to determine the deformed diameter d as a function of other quantities assuming the law

$$f(D, d, \rho, V, E) = 0$$

for some function f . Your answer should look something like

$$d = \text{stuff} \times (\text{some function of dimensionless quantities})$$

3. [10pts] Boltzman studied the electromagnetic energy radiated by substances at different temperatures τ . He believed such radiated energy could only be explained with both quantum mechanics and thermodynamics so should involve the key parameters: Planck's constant \hbar and Boltzmann's constant k where $[k] = \text{joules}/K$ where $K = ^\circ \text{Kelvin}$. Planck's constant has units of joules \times time and the radiation travels at the speed of light c . The energy density \mathcal{E} (joules/L^3) he thought was related to the former parameters. Assume

$$f(\mathcal{E}, \tau, \hbar, k, c) = 0 \quad ,$$

for some function f and then use dimensional analysis to find a formula for \mathcal{E} in terms of dimensional quantities. What power of τ is \mathcal{E} proportional to?

4. [10pts] The population of bacteria in a container of volume V are a fed soluble nutrient at a constant rate F (volume per time). A set of differential equations which models the bacteria concentration $N(t)$ and nutrient concentration $C(t)$ is:

$$\begin{aligned} \frac{dN}{dt} &= \left(\frac{K_{max}C}{K_n + C} \right) N - \frac{FN}{V} \\ \frac{dC}{dt} &= -\alpha \left(\frac{K_{max}C}{K_n + C} \right) N - \frac{FC}{V} + \frac{FC_0}{V} \end{aligned}$$

Here $[N] = ML^{-3}$, $[C] = ML^{-3}$ and the parameters $K_{max}, K_n, F, V, \alpha, C_0$ are constant. We won't worry about why these equations are appropriate but seek to simplify their analysis by first nondimensionalizing them.

- a) Determine the dimensions of $K_{max}, K_n, F, V, \alpha, C_0$ in terms of the fundamental units M, L, T .
- b) Define dimensionless variables

$$n = \frac{N}{N^*} \quad , \quad c = \frac{C}{C^*} \quad , \quad \tau = \frac{t}{t^*}$$

Find N^*, C^* and t^* such that the nondimensionalized system is

$$\begin{aligned} \frac{dn}{d\tau} &= \alpha_1 \left(\frac{c}{1+c} \right) n - n \\ \frac{dc}{d\tau} &= - \left(\frac{c}{1+c} \right) n - c + \alpha_2 \end{aligned}$$

Write out formulae defining α_1, α_2 . Note that the original system had 6 parameters versus 2 for the dimensionless system!

5. [5pts] Consider a physical law involving four quantities:

$$f(q_1, q_2, q_3, q_4) = 0$$

where each is expressible in terms of three fundamental units L_1, L_2, L_3 . The quantity

$$\Pi = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3} q_4^{\alpha_4}$$

is dimensionless only if $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is in the nullspace of a matrix $A \in \mathbb{R}^{3 \times 4}$. We'll suppose that $\dim N(A) = 2$ and that you've found two basis vectors \vec{a} and \vec{b} so that

$$\Pi_a = q_1^{a_1} q_2^{a_2} q_3^{a_3} q_4^{a_4}$$

$$\Pi_b = q_1^{b_1} q_2^{b_2} q_3^{b_3} q_4^{b_4}$$

are dimensionless. Your friend finds a different basis vector:

$$\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b}$$

and associated dimensionless quantity Π_c . Show

$$\Pi_c = \Pi_a^{\lambda_1} \Pi_b^{\lambda_2}$$

QUESTION ONE

$$f(p, m, l, g, \theta) = 0$$

$$(1) \quad \Pi \equiv p^{d_1} m^{d_2} l^{d_3} g^{d_4}$$

then

$$[\Pi] = T^{d_1 - 2d_4} L^{d_3 + d_4} M^{d_2}$$

yields dimension matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

with the following basis vector for $N(A)$.

$$\vec{\alpha} = (1, 0, -\frac{1}{2}, \frac{1}{2}) \quad (\text{not unique})$$

hence

$$\Pi_1 = p \sqrt{\frac{g}{l}}$$

With $\Pi_2 = \theta$ then $\exists F$ s.t. $F(\Pi_1, \Pi_2) = 0$ implying

$$(2) \quad p = \sqrt{\frac{l}{g}} \phi(\theta)$$

Part b) Note that $p_e \sqrt{g_e} = p_m \sqrt{g_m} = \sqrt{l} \phi(\theta)$
so that

$$p_m = \sqrt{\frac{g_e}{g_m}} p_e = \sqrt{\frac{9.8}{1.62}} \approx 2.46 \text{ sec}$$

QUESTION TWO

$$\pi = D^{d_1} d^{d_2} \rho^{d_3} v^{d_4} E^{d_5}$$

Then

$$[\pi] = L^{d_1} L^{d_2} (ML^{-3})^{d_3} (LT^{-1})^{d_4} (ML^{-1}T^{-2})^{d_5}$$

$$[\pi] = L^{d_1+d_2-3d_3+d_4-d_5} M^{d_3+d_5} T^{-d_4-2d_5}$$

Dimension matrix

$$A = \begin{bmatrix} \textcircled{1} & \downarrow & & & \downarrow \\ 1 & -3 & 1 & -1 & \\ 0 & 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & \textcircled{-1} & -2 \end{bmatrix} \quad \begin{array}{l} d_2, d_5 \\ \text{free} \end{array}$$

Two dimensional $N(A)$ with basis vectors

$$\vec{\alpha}_1 = (-1, 1, 0, 0, 0)$$

$$\pi_1 = \frac{d}{D}$$

$$\vec{\alpha}_2 = (0, 0, -1, -2, 1)$$

$$\pi_2 = \frac{E}{\rho v^2}$$

Since $F(\pi_1, \pi_2) = 0 \exists$ some function f s.t.

$$d = D f\left(\frac{E}{\rho v^2}\right)$$

QUESTION THREE

$$\Pi \equiv \varepsilon^{\alpha_1} \tau^{\alpha_2} h^{\alpha_3} k^{\alpha_4} c^{\alpha_5}$$

Then the dimensions are

$$[\Pi] = (ML^{-1}\pi^{-2})^{\alpha_1} (K)^{\alpha_2} (ML^2\pi^{-1})^{\alpha_3} (ML^2\pi^{-2}K^{-1})^{\alpha_4} (L\pi^{-1})^{\alpha_5}$$

yields the following dimension matrix for $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 2 & 1 \\ -2 & 0 & -1 & -2 & -1 \end{bmatrix} \begin{array}{l} M \\ K \\ L \\ \pi \end{array}$$

After row reducing* can find a sole basis vector

$$\vec{\alpha} = (1, -4, 3, -4, 3) \quad \vec{\alpha} \in N(A)$$

Hence a dimensionless group is

$$(1) \quad \Pi_1 = \varepsilon \tau^{-4} h^3 k^{-4} c^3$$

By Buckingham Π -Thm, Π_1 is the root of some function F , i.e. $F(\Pi_1)$. Let α be this root and solve for ε in (1)

$$\varepsilon = \alpha \left(\frac{k^4 \tau^4}{h^3 c^3} \right)$$

Thus ε is proportional to τ^4 .

* extra steps here. many versions of work.

Thus

$$t^* = \frac{V}{F} \quad c^* = K_n$$

and (1) becomes

$$(2) \quad \frac{dn}{dt} = \frac{\alpha_1 n c}{1+c} - n \quad \alpha_1 = \frac{K_{max} V}{F}$$

The equation for $c(t)$ is

$$\frac{dc}{dt} = - \left(\frac{\alpha K_{max} N^* t^*}{K_n} \right) \frac{cn}{1+c} - c + \left(\frac{FC_0 t^*}{VC^*} \right)$$

choose N^* to make one.

Then the choice

$$N^* = \frac{FK_n}{\alpha K_{max} V}$$

yields

$$(3) \quad \frac{dc}{dt} = - \frac{cn}{1+c} - c + \alpha_2$$

where

$$\alpha_2 = \frac{C_0}{K_n}$$

QUESTION FIVE

$$\pi_c = q_1^{c_1} q_2^{c_2} q_3^{c_3} q_4^{c_4}$$

is dimensionless for any $\vec{c} \in N(A)$. If

$$\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b}$$

then

$$\pi_c = q_1^{a_1 \lambda_1 + b_1 \lambda_2} \dots q_4^{a_4 \lambda_1 + b_4 \lambda_2}$$

$$\pi_c = (q_1^{\lambda_1 a_1} \dots q_4^{\lambda_1 a_4}) (q_1^{\lambda_2 b_1} \dots q_4^{\lambda_2 b_4})$$

$$\pi_c = (q_1^{a_1} \dots q_4^{a_4})^{\lambda_1} (q_1^{b_1} \dots q_4^{b_4})^{\lambda_2}$$

$$\pi_c = \pi_a^{\lambda_1} \cdot \pi_b^{\lambda_2}$$