

## Math 450 (2009) – Homework 3

Due: October 16, 2009.

NAME: \_\_\_\_\_

1. [6pts] Let  $x = \bar{x}(\epsilon)$  be the solution of

$$f(x, \epsilon) = x - \sqrt{4 + \epsilon x} = 0 \quad , \quad 0 < \epsilon \ll 1$$

a) Assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3)$$

and then determine  $x_0$  and  $x_1$  only. In your calculations you may need the Binomial Theorem

$$(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \dots$$

where here  $p = \frac{1}{2}$  and  $z = \frac{\epsilon x}{4}$ .

b) Find an exact formula for  $\bar{x}(\epsilon)$  and then show its Taylor series

$$\bar{x}(\epsilon) = \bar{x}(0) + \bar{x}'(0)\epsilon + \frac{1}{2!}\bar{x}''(0)\epsilon^2 + O(\epsilon^3)$$

agrees with your solution derived in part a), i.e., does  $\bar{x}(0) = x_0$ , etc. Here, also explain why there is only one root.

2. [6pts] One can find regular expansions for solutions to coupled algebraic equations.

Consider the coupled system

$$2x - y = \epsilon(x - y)$$

$$2x + y - 1 = \epsilon xy^2$$

Assume the exact solution  $(\bar{x}(\epsilon), \bar{y}(\epsilon))$  can be expanded as follows:

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

$$\bar{y}(\epsilon) = y_0 + y_1\epsilon + O(\epsilon^2)$$

and then determine  $x_0, x_1, y_0$  and  $y_1$ .

3. [8pts] Consider the initial value problem:

$$y'' - y = \epsilon \log(y) \quad , \quad y(0) = 1 \quad , \quad y'(0) = 1$$

where  $0 < \epsilon \ll 1$ . Find  $y_0(t)$  and  $y_1(t)$  in the assumed expansion of the solution  $y$ :

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)$$

Here  $\log$  is base  $e$  and you may use the expansion:

$$\log(a + \epsilon b) = \log(a) + \frac{b}{a}\epsilon + O(\epsilon^2)$$

4. [6pts] Regular perturbation techniques can be applied to approximate the solution of two-point Boundary Value Problems (BVP). Let  $y(x)$  be the solution of the nonlinear BVP

$$y' y'' = \epsilon x (y')^2 \quad , \quad y(0) = 0 \quad , \quad y(1) = 1$$

where  $0 < \epsilon \ll 1$ . Notice that the values of  $y(x, \epsilon)$  are specified at the boundaries  $x = 0$  and  $x = 1$ .

Find  $y_0(x)$  and  $y_1(x)$  in the assumed expansion of the solution  $y$ :

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

5. [4pts] One can find regular expansions for solutions to linear systems. For example, let  $A, B \in \mathbb{R}^{2 \times 2}$  be  $\epsilon$  independent constant matrices. One might be interested in finding the vector solution  $\mathbf{x}(\epsilon)$  of the perturbed matrix equation:

$$(A + \epsilon B)\mathbf{x} = \mathbf{b} \quad , \quad 0 < \epsilon \ll 1$$

and  $\mathbf{b} = (b_1, b_2)^T$  is a constant vector. Assume the (vector) expansion

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + O(\epsilon^3) = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \dots$$

Show that

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 - \epsilon (A^{-1}B) \mathbf{x}_0 + \epsilon^2 (A^{-1}B)^2 \mathbf{x}_0 + O(\epsilon^3)$$

where  $\mathbf{x}_0$  is the solution of the unperturbed ( $\epsilon = 0$ ) problem  $A\mathbf{x}_0 = \mathbf{b}$  and  $A^{-1}$  is the inverse of the matrix  $A$ .