

Math 450 (2011) – Homework 3

Due: October 21, 2011.

NAME: _____

1. [10pts] Let $x = \bar{x}(\epsilon)$ be the root of $f(x, \epsilon) = 0$ and assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3)$$

This assumption can be verified with the implicit function theorem. Find x_0, x_1 and x_2 for the following functions:

a) $f(x, \epsilon) = x^3 - 6\epsilon x^2 + 3\epsilon^2 - 1$

b) $f(x, \epsilon) = \exp\left(\frac{1}{4}x\right) - \frac{4}{\sqrt{4+\epsilon x}}$

For the latter you may use the expansion:

$$(1+z)^{-1/2} = 1 - \frac{1}{2}z + \frac{3}{8}z^2 + O(z^3), \quad z = \frac{\epsilon x}{4}$$

Organize your work by summarizing the $O(1), O(\epsilon)$ and $O(\epsilon^2)$ problems.

2. [5pts] One can find regular expansions for solutions to coupled algebraic equations.

Consider the coupled system

$$x + y = \epsilon(x + 6y)$$

$$xy^2 - 1 = \epsilon x^2$$

Assume the exact solution $(\bar{x}(\epsilon), \bar{y}(\epsilon))$ can be expanded as follows:

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

$$\bar{y}(\epsilon) = y_0 + y_1\epsilon + O(\epsilon^2)$$

and then determine x_0, x_1, y_0 and y_1 .

3. [5pts] Consider the initial value problem:

$$y'' + y = \frac{\epsilon}{\sqrt{1-y(t)^2}}, \quad y(0) = \epsilon, \quad y'(0) = 1$$

where $0 < \epsilon \ll 1$. Find $y_0(t)$ and $y_1(t)$ in the assumed expansion of the solution y :

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)$$

4. [8pts] Regular perturbation techniques can be applied to approximate the solution of two-point Boundary Value Problems (BVP). Let $y(x)$ be the solution of the nonlinear BVP

$$y'' - \frac{18}{y'} = 21\epsilon(y - 1) \quad , \quad y(0) = 1 \quad , \quad y(1) = 5$$

where $0 < \epsilon \ll 1$. Notice that the values of $y(x, \epsilon)$ are specified at the boundaries $x = 0$ and $x = 1$. We shall assume y has the expansion:

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

a) Clearly define and solve the $O(1)$ problem for $y_0(x)$. In particular show that

$$y_0(x) = 4x^{3/2} + 1$$

b) Derive and then solve the $O(\epsilon)$ problem for $y_1(x)$. Note that constants are solutions to the homogeneous problem for y_1 .

5. [7pts] One can find regular expansions for perturbed linear systems. For example, define

$$A = \begin{bmatrix} 1 & 2 + \epsilon \\ 2 - \epsilon & 1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 2 - \epsilon \\ 3 \end{pmatrix}$$

We seek a regular approximation for the solution $\mathbf{x}(\epsilon)$ of

$$A(\epsilon) \mathbf{x}(\epsilon) = \mathbf{b}(\epsilon) \tag{1}$$

Use the expansion

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + O(\epsilon^3) = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \dots$$

in (1). By collecting powers of ϵ , arrive at $O(1)$, $O(\epsilon)$ and $O(\epsilon^2)$ problems. State these clearly. Then, solve these problems sequentially to find $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$. To do all this in an organized way let

$$A = A_0 + \epsilon A_1$$

$$\mathbf{b} = \mathbf{b}_0 + \epsilon \mathbf{b}_1$$

and do the calculations for a general A_0, A_1, \mathbf{b}_0 , etc. Then, at the very end use the specific values defined above.