

## Math 450 (2009) – Homework 3

Due: October 16, 2009.

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1. [6pts] Let  $x = \bar{x}(\epsilon)$  be the solution of

$$f(x, \epsilon) = x - \sqrt{4 + \epsilon x} = 0 \quad , \quad 0 < \epsilon \ll 1$$

a) Assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3)$$

and then determine  $x_0$  and  $x_1$  only. In your calculations you may need the Binomial Theorem

$$(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \dots$$

where here  $p = \frac{1}{2}$  and  $z = \frac{\epsilon x}{4}$ .

b) Find an exact formula for  $\bar{x}(\epsilon)$  and then show its Taylor series

$$\bar{x}(\epsilon) = \bar{x}(0) + \bar{x}'(0)\epsilon + \frac{1}{2!}\bar{x}''(0)\epsilon^2 + O(\epsilon^3)$$

agrees with your solution derived in part a), i.e., does  $\bar{x}(0) = x_0$ , etc. Here, also explain why there is only one root.

2. [6pts] One can find regular expansions for solutions to coupled algebraic equations. Consider the coupled system

$$2x - y = \epsilon(x - y)$$

$$2x + y - 1 = \epsilon xy^2$$

Assume the exact solution  $(\bar{x}(\epsilon), \bar{y}(\epsilon))$  can be expanded as follows:

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

$$\bar{y}(\epsilon) = y_0 + y_1\epsilon + O(\epsilon^2)$$

and then determine  $x_0, x_1, y_0$  and  $y_1$ .

3. [8pts] Consider the initial value problem:

$$y'' - y = \epsilon \log(y) \quad , \quad y(0) = 1 \quad , \quad y'(0) = 1$$

where  $0 < \epsilon \ll 1$ . Find  $y_0(t)$  and  $y_1(t)$  in the assumed expansion of the solution  $y$ :

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)$$

Here  $\log$  is base  $e$  and you may use the expansion:

$$\log(a + \epsilon b) = \log(a) + \frac{b}{a}\epsilon + O(\epsilon^2)$$

4. [6pts] Regular perturbation techniques can be applied to approximate the solution of two-point Boundary Value Problems (BVP). Let  $y(x)$  be the solution of the nonlinear BVP

$$y' y'' = \epsilon x (y')^2 \quad , \quad y(0) = 0 \quad , \quad y(1) = 1$$

where  $0 < \epsilon \ll 1$ . Notice that the values of  $y(x, \epsilon)$  are specified at the boundaries  $x = 0$  and  $x = 1$ .

Find  $y_0(x)$  and  $y_1(x)$  in the assumed expansion of the solution  $y$ :

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

5. [4pts] One can find regular expansions for solutions to linear systems. For example, let  $A, B \in \mathbb{R}^{2 \times 2}$  be  $\epsilon$  independent constant matrices. One might be interested in finding the vector solution  $\mathbf{x}(\epsilon)$  of the perturbed matrix equation:

$$(A + \epsilon B)\mathbf{x} = \mathbf{b} \quad , \quad 0 < \epsilon \ll 1$$

and  $\mathbf{b} = (b_1, b_2)^T$  is a constant vector. Assume the (vector) expansion

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + O(\epsilon^3) = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \dots$$

Show that

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 - \epsilon (A^{-1}B) \mathbf{x}_0 + \epsilon^2 (A^{-1}B)^2 \mathbf{x}_0 + O(\epsilon^3)$$

where  $\mathbf{x}_0$  is the solution of the unperturbed ( $\epsilon = 0$ ) problem  $A\mathbf{x}_0 = \mathbf{b}$  and  $A^{-1}$  is the inverse of the matrix  $A$ .



## QUESTION TWO

Use expansions

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$$

in system of equations. Then expand and collect like powers

$$(1) \quad 2(x_0 + \varepsilon x_1 + \dots) - (y_0 + \varepsilon y_1 + \dots) = \varepsilon(x_0 - y_0 + \dots)$$

$$(2) \quad 2(x_0 + \varepsilon x_1 + \dots) + (y_0 + \varepsilon y_1 + \dots) - 1 = \varepsilon(x_0 + \varepsilon x_1 + \dots)(y_0 + \varepsilon y_1 + \dots)^2$$

Collect like powers

$$(2x_0 - y_0) + \varepsilon(2x_1 - y_1) = \varepsilon(x_0 - y_0) + O(\varepsilon^2)$$

$$(2x_0 + y_0 - 1) + \varepsilon(2x_1 + y_1) = \varepsilon x_0 y_0^2 + O(\varepsilon^2)$$

Yields two systems

$$\left. \begin{aligned} 2x_0 - y_0 &= 0 \\ 2x_0 + y_0 - 1 &= 0 \end{aligned} \right\} x_0 = \frac{1}{4}, y_0 = \frac{1}{2}$$

and

$$\left. \begin{aligned} 2x_1 - y_1 &= x_0 - y_0 \\ 2x_1 + y_1 &= x_0 y_0^2 \end{aligned} \right\} x_1 = -\frac{3}{64}, y_1 = \frac{5}{32}$$

Conclude

$$x = \frac{1}{4} - \frac{3}{64} \varepsilon + O(\varepsilon^2)$$

$$y = \frac{1}{2} + \frac{5}{32} \varepsilon + O(\varepsilon^2)$$

### QUESTION THREE

Assume

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2) \quad 0 < \epsilon \ll 1$$

Initial Conditions

$$y(0, \epsilon) = y_0(0) + \epsilon y_1(0) + \dots = 1$$

$$y'(0, \epsilon) = y_0'(0) + \epsilon y_1'(0) + \dots = 1$$

Hence

$$y_0(0) = y_0'(0) = 1 \quad y_1(0) = y_1'(0) = 0$$

Now expand differential equation and collect like powers of  $\epsilon$

$$(y_0'' + \epsilon y_1'' + \dots) - (y_0 + \epsilon y_1 + \dots) = \epsilon \log(y_0 + \epsilon y_1 + \dots)$$

$$(y_0'' - y_0) + \epsilon (y_1'' - y_1) = \epsilon \log y_0 + O(\epsilon^2)$$

Hence

$$(1) \quad y_0'' - y_0 = 0 \quad y_0(0) = 1 \quad y_0'(0) = 1$$

$$(2) \quad y_1'' - y_1 = \log(y_0) \quad y_1(0) = 0 \quad y_1'(0) = 0$$

The solution of (1) is  $y_0 = e^x$  hence  $\log(y_0) = x$ .  
Then (2) is

$$y_1'' - y_1 = x$$

whose soln is (after some work)

$$y_1(x) = \frac{1}{2}(e^x - e^{-x}) - x = \sinh x - x$$

### QUESTION FOUR

Assume  $y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + \dots$  Expand Diff Eq.

$$(y_0' + \epsilon y_1' + \dots)(y_0'' + \epsilon y_1'' + \dots) = \epsilon x (y_0' + \epsilon y_1' + \dots)^2$$

$$y_0' y_0'' + \epsilon (y_0' y_1'' + y_0'' y_1') = \epsilon x (y_0')^2 + O(\epsilon^2)$$

Equating powers of  $\epsilon$  and considering boundary values

$$(1) \quad y_0' y_0'' = 0 \quad y_0(0) = 0 \quad y_0(1) = 1$$

$$(2) \quad y_0' y_1'' + y_0'' y_1' = x (y_0')^2 \quad y_1(0) = 0 \quad y_1(1) = 0$$

Noting  $\frac{d}{dx} (y_0')^2 = 2 y_0' y_0''$  we see the general soln for  $y_0$  is

$$y_0(x) = Ax + B$$

Apply BC,  $y_0(0) = B = 0$  and  $y_0(1) = A = 1$ , hence

$$(3) \quad y_0(x) = x$$

Using (3) in (2) we get the simplified problem

$$y_1'' = x$$

$$y_1 = \frac{1}{6} x^3 + Ax + B$$

Apply BC,  $y_1(0) = B = 0$  and  $y_1(1) = \frac{1}{6} + A = 0$  so

$$(4) \quad y_1(x) = \frac{1}{6} x^3 - \frac{1}{6} x$$

## QUESTION FIVE

Assume

$$\vec{x}(\epsilon) = \vec{x}_0 + \epsilon \vec{x}_1 + \epsilon^2 \vec{x}_2 + \dots$$

Substitute into  $(A + \epsilon B) \vec{x} = \vec{b}$

$$(A + \epsilon B)(\vec{x}_0 + \epsilon \vec{x}_1 + \epsilon^2 \vec{x}_2 + \dots) = \vec{b}$$

Expand and collect like powers of  $\epsilon$

$$\begin{aligned} A \vec{x}_0 + \epsilon A \vec{x}_1 + \epsilon^2 A \vec{x}_2 + \dots &= \vec{b} \\ + \epsilon B \vec{x}_0 + \epsilon^2 B \vec{x}_1 + \dots & \end{aligned}$$

Thus

$$O(1) \quad A \vec{x}_0 = \vec{b} \quad (\vec{x}_0 = A^{-1} \vec{b})$$

$$O(\epsilon) \quad A \vec{x}_1 = -B \vec{x}_0$$

$$O(\epsilon^2) \quad A \vec{x}_2 = -B \vec{x}_1$$

Since  $\vec{x}_0 = A^{-1} \vec{b}$  then  $O(\epsilon)$  eqn yields

$$\vec{x}_1 = -A^{-1} B \vec{x}_0$$

and  $O(\epsilon^2)$  yields

$$\vec{x}_2 = -A^{-1} B \vec{x}_1 = + (A^{-1} B) \overbrace{(A^{-1} B) \vec{x}_0}^{-\vec{x}_1} = (A^{-1} B)^2 \vec{x}_0$$

to conclude

$$\vec{x}(\epsilon) = \vec{x}_0 - A^{-1} B \vec{x}_0 + (A^{-1} B)^2 \vec{x}_0$$

where  $\vec{x}_0$  is the leading order soln  $\vec{x}_0 = A^{-1} \vec{b}$ .