

Math 450 (2011) – Homework 3

Due: October 21, 2011.

NAME: _____

1. [10pts] Let $x = \bar{x}(\epsilon)$ be the root of $f(x, \epsilon) = 0$ and assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3)$$

This assumption can be verified with the implicit function theorem. Find x_0, x_1 and x_2 for the following functions:

a) $f(x, \epsilon) = x^3 - 6\epsilon x^2 + 3\epsilon^2 - 1$

b) $f(x, \epsilon) = \exp\left(\frac{1}{4}x\right) - \frac{4}{\sqrt{4+\epsilon x}}$

For the latter you may use the expansion:

$$(1+z)^{-1/2} = 1 - \frac{1}{2}z + \frac{3}{8}z^2 + O(z^3) \quad , \quad z = \frac{\epsilon x}{4}$$

Organize your work by summarizing the $O(1), O(\epsilon)$ and $O(\epsilon^2)$ problems.

2. [5pts] One can find regular expansions for solutions to coupled algebraic equations.

Consider the coupled system

$$\begin{aligned} x + y &= \epsilon(x + 6y) \\ xy^2 - 1 &= \epsilon x^2 \end{aligned}$$

Assume the exact solution $(\bar{x}(\epsilon), \bar{y}(\epsilon))$ can be expanded as follows:

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

$$\bar{y}(\epsilon) = y_0 + y_1\epsilon + O(\epsilon^2)$$

and then determine x_0, x_1, y_0 and y_1 .

3. [5pts] Consider the initial value problem:

$$y'' + y = \frac{\epsilon}{\sqrt{1 - y(t)^2}} \quad , \quad y(0) = \epsilon \quad , \quad y'(0) = 1$$

where $0 < \epsilon \ll 1$. Find $y_0(t)$ and $y_1(t)$ in the assumed expansion of the solution y :

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)$$

4. [8pts] Regular perturbation techniques can be applied to approximate the solution of two-point Boundary Value Problems (BVP). Let $y(x)$ be the solution of the nonlinear BVP

$$y'' - \frac{18}{y'} = 21\epsilon(y - 1) \quad , \quad y(0) = 1 \quad , \quad y(1) = 5$$

where $0 < \epsilon \ll 1$. Notice that the values of $y(x, \epsilon)$ are specified at the boundaries $x = 0$ and $x = 1$. We shall assume y has the expansion:

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

a) Clearly define and solve the $O(1)$ problem for $y_0(x)$. In particular show that

$$y_0(x) = 4x^{3/2} + 1$$

b) Derive and then solve the $O(\epsilon)$ problem for $y_1(x)$. Note that constants are solutions to the homogeneous problem for y_1 .

5. [7pts] One can find regular expansions for perturbed linear systems. For example, define

$$A = \begin{bmatrix} 1 & 2 + \epsilon \\ 2 - \epsilon & 1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 2 - \epsilon \\ 3 \end{pmatrix}$$

We seek a regular approximation for the solution $\mathbf{x}(\epsilon)$ of

$$A(\epsilon) \mathbf{x}(\epsilon) = \mathbf{b}(\epsilon) \tag{1}$$

Use the expansion

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + O(\epsilon^3) = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \dots$$

in (1). By collecting powers of ϵ , arrive at $O(1)$, $O(\epsilon)$ and $O(\epsilon^2)$ problems. State these clearly. Then, solve these problems sequentially to find $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$. To do all this in an organized way let

$$A = A_0 + \epsilon A_1$$

$$\mathbf{b} = \mathbf{b}_0 + \epsilon \mathbf{b}_1$$

and do the calculations for a general A_0, A_1, \mathbf{b}_0 , etc. Then, at the very end use the specific values defined above.

QUESTION ONE a)

$$f = (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^3 - 6\epsilon (x_0 + \epsilon x_1 + \dots)^2 + 3\epsilon^2 - 1$$

$$f = (x_0^3 + 3x_0^2 \underset{\uparrow}{x_1} \epsilon + (3x_2 x_0^2 + 3x_1^2 x_0) \underset{\uparrow}{\epsilon^2}) - 6\epsilon (x_0^2 + 2x_0 \underset{\uparrow}{x_1} \epsilon + \dots) + 3\epsilon^2 - 1$$

Collect like powers as indicated

$$x_0^3 - 1 = 0 \quad O(1)$$

$$3x_0^2 \underline{x_1} - 6x_0^2 = 0 \quad O(\epsilon)$$

$$x_0^2 \underline{x_2} + x_1^2 x_0 - 4x_0 x_1 + 1 = 0 \quad O(\epsilon^2)$$

Solve sequentially for x_k

$$x_0 = 1 \quad x_1 = 2 \quad x_2 = 3$$

Hence

$$\bar{x}(\epsilon) = 1 + 2\epsilon + 3\epsilon^2 + O(\epsilon^3)$$

Remark: To expand $\bar{x}(\epsilon)^3$ let $\bar{x} = x_0 + \epsilon z$
where $z = x_1 + \epsilon x_2 + \dots$

$$(x_0 + \epsilon z)^3 = x_0^3 + 3x_0^2 z \epsilon + 3x_0 z^2 \epsilon^2 + O(\epsilon^3)$$

QUESTION ONE b)

$$\frac{4}{\sqrt{4+\epsilon x}} = 2(1+z)^{-\frac{1}{2}} \quad z = \frac{\epsilon x}{4}$$

$$= 2 \left\{ 1 - \frac{1}{2} \frac{\epsilon}{4} (\overset{\downarrow}{x_0} + \overset{\downarrow}{\epsilon x_1} + \dots) + \frac{3}{8} \frac{\epsilon^2}{16} (\overset{\downarrow}{x_0} + \epsilon x_1 + \dots)^2 + \dots \right\}$$

$$= 2 - \frac{1}{4} x_0 \epsilon + \left(-\frac{1}{4} x_1 + \frac{3}{64} x_0^2 \right) \epsilon^2 + O(\epsilon^3)$$

Next expand the exponential

$$e^{\frac{1}{4}x} = e^{\frac{1}{4}(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots)}$$

$$= e^{\frac{1}{4}x_0} e^{\frac{1}{4}(\epsilon x_1 + \epsilon^2 x_2 + \dots)}$$

$$= e^{\frac{1}{4}x_0} \left(1 + \frac{1}{4} (\overset{\downarrow}{\epsilon x_1} + \overset{\downarrow}{\epsilon^2 x_2} + \dots) + \frac{1}{2!} \frac{1}{16} (\overset{\downarrow}{\epsilon x_1} + \dots)^2 + \dots \right)$$

$$= e^{\frac{1}{4}x_0} \left(1 + \frac{1}{4} x_1 \epsilon + \left(\frac{1}{4} x_2 + \frac{1}{32} x_1^2 \right) \epsilon^2 + \dots \right)$$

Results in

$$e^{\frac{1}{4}x_0} - 2 = 0 \quad O(1)$$

$$e^{\frac{1}{4}x_0} x_1 + x_0 = 0 \quad O(\epsilon)$$

$$e^{\frac{1}{4}x_0} (16x_2 + 2x_1^2) - 3x_0^2 + 16x_1 = 0 \quad O(\epsilon^2)$$

Solving yields

$$x_0 = 4 \ln 2 \quad x_1 = -2 \ln 2 \quad x_2 = \ln(2)^2 + \ln 2$$

In the algebra note

$$e^{\frac{1}{4}x_0} = 2 \text{ helps for } O(\epsilon), O(\epsilon^2) \text{ problems}$$

QUESTION TWO

$$(x_0 + \epsilon x_1 + \dots) + (y_0 + \epsilon y_1 + \dots) = \epsilon(x_0 + 6y_0) + O(\epsilon^2)$$

$$(x_0 + \epsilon x_1 + \dots)(y_0 + \epsilon y_1 + \dots)^2 - 1 = \epsilon x_0^2 + O(\epsilon^2)$$

Collecting like powers of ϵ

$O(1)$ problem

$$\begin{array}{r} x_0 + y_0 = 0 \\ x_0 y_0^2 - 1 = 0 \end{array}$$

$O(\epsilon)$ problem

$$\begin{array}{r} x_1 + y_1 = x_0 + 6y_0 \\ y_0^2 x_1 + 2x_0 y_0 y_1 = x_0^2 y_0 \end{array}$$

Solution of $O(1)$ problem is

$$x_0 = 1 \quad y_0 = -1$$

Used in $O(\epsilon)$ problem yields

$$\begin{array}{r} x_1 + y_1 = -5 \\ x_1 - 2y_1 = 1 \end{array} \Rightarrow x_1 = -3, y_1 = -2$$

Conclude

$$x = 1 - 3\epsilon + O(\epsilon^2)$$

$$y = -1 - 2\epsilon + O(\epsilon^2)$$

QUESTION THREE

Since we need only retain $O(\epsilon)$ terms, i.e. y_0, y_1 and since

$$\frac{1}{\sqrt{1-y^2}} = \frac{1}{\sqrt{1-y_0^2}} + O(\epsilon)$$

we easily deduce

$$O(1) \quad y_0'' + y_0 = 0$$

$$y_0(0) = 0$$

$$y_0'(0) = 1$$

$$O(\epsilon) \quad y_1'' + y_1 = \frac{1}{\sqrt{1-y_0^2}}$$

$$y_1(0) = 1$$

$$y_1'(0) = 0$$

Solution of $O(\epsilon)$ problem is

$$y_0(x) = \sin x$$

Since $1 - y_0^2 = \cos^2 x$ the $O(\epsilon)$ problem becomes

$$y_1'' + y_1 = \sec x$$

$$y_1(0) = 1$$

$$y_1'(0) = 0$$

The particular soln found by variation of parameters is $y_1^p(x) = x \sin x + \cos x \ln(\cos x)$.
Ultimately

$$y_0(x) = \sin x$$

$$y_1(x) = \cos x + x \sin x + \cos x \ln(\cos x)$$

QUESTION FOUR

First rewrite as:

$$y'y'' - 18 = 21\epsilon y'(y-1)$$

Then $y = y_0 + \epsilon y_1 + O(\epsilon^2)$

$$(y_0' + \epsilon y_1' + \dots)(y_0'' + \epsilon y_1'' + \dots) - 18 = 21\epsilon (y_0' + \epsilon y_1' + \dots)(y_0 - 1 + \dots)$$

Collect powers of ϵ yields the problems

$$O(1) \quad y_0' y_0'' = 18 \quad y_0(0) = 1, \quad y_0(1) = 5$$

$$O(\epsilon) \quad y_0' y_1'' + y_0'' y_1' = 21(y_0 - 1)y_0' \quad y_1(0) = 0 \quad y_1(1) = 0$$

Soln of $O(1)$ problem

$$\frac{d}{dx} \left(\frac{1}{2} y_0'^2 \right) = 18$$

$$y_0'^2 = 36x + c_1$$

$$y_0' = \sqrt{36x + c_1}$$

By renaming the arbitrary const c_1 and integrating in x we get

$$y_0(x) = 4(x + c_1)^{3/2} + c_2$$

which satisfies the B.C. for $c_1 = 0, c_2 = 1$

$$y_0(x) = 4x^{3/2} + 1$$

Solution of $O(\epsilon)$ problem

$$y_0' y_1'' + y_0'' y_1' = 21 (y_0^{-1}) y_0'$$
$$6\sqrt{x} y_1'' + \frac{3}{\sqrt{x}} y_1' = 504 x^2$$

Is linear in $v(x) \equiv y_1'$. In standard form

$$v' + \frac{1}{2x} v = 84 x^{3/2}$$

whose general soln is

$$y_1'(x) = v(x) = 28 x^{5/2} + C_1 x^{-1/2}$$

Integrate

$$y_1(x) = 8 x^{7/2} + 2C_1 \sqrt{x} + C_2$$

Use boundary conditions

$$y_1(0) = C_2 = 0$$

$$y_1(1) = 8 + 2C_1 + C_2 = 0$$

hence $C_1 = -4, C_2 = 0$.

$$y_1(x) = 8 \left(x^{7/2} - \sqrt{x} \right)$$

QUESTION FIVE

$$(A_0 + \epsilon A_1)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) = b_0 + \epsilon b_1$$

Collect like powers of ϵ :

$$A_0 x_0 = b_0 \quad O(1)$$

$$A_0 x_1 = b_1 - A_1 x_0 \quad O(\epsilon)$$

$$A_0 x_2 = -A_1 x_1 \quad O(\epsilon^2)$$

For our specific problem

$$A_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad b_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad b_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Inverse of A_0 easily found and $O(1)$ soln is:

$$x_0 = \begin{pmatrix} 4/3 \\ 1/3 \end{pmatrix}$$

Hence $O(\epsilon)$ problem becomes

$$A_0 x_1 = \begin{pmatrix} -4/3 \\ 4/3 \end{pmatrix}$$

Solve for x_1 , and then $O(\epsilon^2)$ problem

$$x_0 = \begin{pmatrix} 4/3 \\ 1/3 \end{pmatrix} \quad x_1 = \begin{pmatrix} 4/3 \\ -4/3 \end{pmatrix} \quad x_2 = \begin{pmatrix} 4/9 \\ 4/9 \end{pmatrix}$$