

Math 450 (2009) – Homework 4

Due: Wednesday, November 9, 2011.

NAME: _____

1. [9 pts] Let $y(t, \epsilon)$ be the solution of the initial value problem

$$\begin{aligned}y'' + y &= \epsilon y(y'')^2, \quad 0 < \epsilon \ll 1 \\y(0) &= 0, \quad y'(0) = 1\end{aligned}$$

where $()'$ denotes differentiation with respect to t . Assume

$$\begin{aligned}y(t, \epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon)t \equiv (1 + \omega_1\epsilon + O(\epsilon^2))t\end{aligned}$$

where $y_k(\tau)$ are 2π -periodic in τ for appropriate choices of ω_k for $k \geq 1$. Use Poincaré-Lindstedt's method to find $y_0(\tau)$, $y_1(\tau)$ and the corrected period of the oscillation, i.e., T_0 and T_1 in the exact period (in the original time t):

$$T(\epsilon) = T_0 + \epsilon T_1 + O(\epsilon^2)$$

You will need to look up appropriate trigonometric identities to complete the problem.

2. [6 pts] Prove that as $\epsilon \rightarrow 0^+$ the following are true:

$$\begin{aligned}\ln(1 + \epsilon^n) &\ll \epsilon^{n-1}, \quad \forall n > 0 \\ \epsilon^\epsilon &= O(1) \\ \sqrt{1 - \cos \epsilon} &= O(\tan \epsilon)\end{aligned}$$

All involve limits of the ratios of the functions. Some of these limits are nontrivial. In one taking the log of the ratio helps. Another hint: $\sqrt{1 + \cos \epsilon}$

3. [5pts] An asymptotic sequence $\{\phi_n(\epsilon)\}$ is defined by $\phi_n(\epsilon) = (\ln(1 + \epsilon))^n$ for $n \geq 0$ noting $\phi_0 = 1$. Find constants a_0, a_1 and a_2 such that

$$f(\epsilon) \equiv \sqrt{4 - 2\epsilon} \sim a_0\phi_0(\epsilon) + a_1\phi_1(\epsilon) + a_2\phi_2(\epsilon) + O(\phi_3) \quad \text{as } \epsilon \rightarrow 0$$

See the posted sample homework to guide you on this. Also, note $\phi_n(\epsilon) = O(\epsilon^n)$.

4. [10 pts] Consider the equation

$$f(x, \epsilon) = \epsilon x^4 - \sqrt{1+x} + 2 = 0$$

One can prove that for sufficiently small ϵ this equation has two roots only one of which is regular in ϵ .

a) Compute x_0, x_1 in the regular expansion of the root $\bar{x}(\epsilon)$

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

b) For the singular root $\bar{X}(\epsilon)$, determine X_0, X_1 and α in the expansion

$$\bar{X}(\epsilon) = \frac{1}{\epsilon^\alpha} \left(X_0 + \delta X_1 + O(\delta^2) \right) \quad , \quad \alpha > 0 \quad , \quad X_0 \neq 0$$

for an appropriate function $\delta(\epsilon) \ll 1$. The dominant balance will be between the first two terms. Also, during the process you will need to expand something like (not exactly the same as)

$$f(\delta) = \sqrt{X_0 + X_1\delta + \dots} \quad , \quad \delta \ll 1$$

The best way to think of this is

$$f(\delta) = f(0) + f'(0)\delta + O(\delta^2)$$

Remember the first step is to let

$$x = \frac{X}{\epsilon^\alpha}$$

find α and then expand X in some δ .