

Math 450 (2009) – Homework 4

Due: November 6, 2009.

NAME: _____

1. [10 pts] Let $y(t, \epsilon)$ be the solution of the initial value problem

$$\begin{aligned}y'' + y &= \epsilon y(y')^2, \quad 0 < \epsilon \ll 1 \\y(0) &= 0, \quad y'(0) = 1\end{aligned}$$

where $()'$ denotes differentiation with respect to t . Assume

$$\begin{aligned}y(t, \epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon) \equiv 1 + \omega_1 \epsilon + O(\epsilon^2)\end{aligned}$$

where $y_k(\tau)$ are 2π -periodic in τ for appropriate choices of ω_k for $k \geq 1$. Use Poincaré-Lindstedt's method to find $y_0(\tau)$, $y_1(\tau)$ and the corrected period of the oscillation, i.e., T_0 and T_1 in the exact period (in the original time t):

$$T(\epsilon) = T_0 + \epsilon T_1 + O(\epsilon^2)$$

You will need to look up appropriate trigonometric identities to complete the problem.

2. [9 pts] Prove that as $\epsilon \rightarrow 0^+$ the following are true:

$$\begin{aligned}e^{-1/\epsilon} &\ll \epsilon^n, \quad \forall n > 0 \\ \int_0^\epsilon f(x) dx &= O(\epsilon) \\ \log(\epsilon) &\ll \frac{1}{1 - \cos(\epsilon)}\end{aligned}$$

For the first, consider the log of the ratio to make the conclusion. The middle can be proved using the Fundamental Theorem of Calculus and the last can be shown using L'Hospital's rule (though there is a simpler way).

3. [6pts] An asymptotic sequence $\{\phi_n(\epsilon)\}$ is defined by $\phi_n(\epsilon) = \sin^n \epsilon$ for $n \geq 0$ noting $\phi_0 = 1$. Find constants a_0, a_1, a_2 and a_3 such that

$$f(\epsilon) \equiv \sqrt{1 - 4\epsilon} \sim a_0 \phi_0(\epsilon) + a_1 \phi_1(\epsilon) + a_2 \phi_2(\epsilon) + a_3 \phi_3(\epsilon) + O(\phi_4) \quad \text{as } \epsilon \rightarrow 0$$

Hint: expand both sides in powers of ϵ .

4. [10 pts] Consider the equation

$$f(x, \epsilon) = \epsilon x^2 - \sqrt{x} + 1 = 0$$

Using calculus one can prove that there are exactly two positive roots to the above equation. If you plot ϵx^2 and $\sqrt{x} - 1$ you can quickly see that for ϵ small, one root is $O(1)$ and the other is singular in ϵ .

a) Compute x_0, x_1 in the regular expansion

$$\bar{x}_-(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

b) For the singular root, determine X_0, X_1 and α in the expansion

$$\bar{x}_+(\epsilon) = \frac{1}{\epsilon^\alpha} \left(X_0 + \delta X_1 + O(\delta^2) \right) \quad , \quad \alpha > 0$$

for an appropriate function $\delta(\epsilon) \ll 1$.

5. [10 pts] Find the leading inner and outer solutions $y_0(x)$ and $Y_0(X)$ of the boundary value problem

$$\begin{aligned} \epsilon y'' + y' + y^2 &= 0 \quad , \quad x \in (0, 1) \\ y(0) &= \frac{1}{4} \quad , \\ y(1) &= \frac{1}{2} \quad , \end{aligned}$$

and then a uniformly valid approximation $y_u(x, \epsilon)$.