<u>Homework 5</u>

QUESTION ONE

$$(1) \qquad y'' + y = \varepsilon y (y'')^2$$

$$(2) \qquad y(0) = 0 \qquad y'(0) = 1$$

Expansion

$$y(t, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2)$$

$$\tau = (1 + \omega_1 \varepsilon + O(\varepsilon^2))\, t$$

<u>Initial conditions</u>

$$y(0, \varepsilon) = y_0(0) + \varepsilon y_1(0) + O(\varepsilon^2) = 0$$

$$y'(0, \varepsilon) = y_0'(0) + \varepsilon (y_1'(0) + \omega_1 y_0'(0)) + O(\varepsilon^2) = 1$$

conclude

$$(3) \qquad y_0(0) = 0 \qquad y_0'(0) = 1$$

$$(4) \qquad y_1(0) = 0 \qquad y_1'(0) = -\omega_1$$

<u>Expand Differential Equation</u>

$$(1 + \omega_1 \varepsilon + \cdots)^2 (y_0'' + \varepsilon y_1'' + \cdots)$$

$$+ (y_0 + \varepsilon y_1 + \cdots) = \varepsilon (y_0 + \varepsilon y_1 + \cdots)(1 + \omega_1 \varepsilon + \cdots)^4 (y_0'' + \varepsilon y_1'' + \cdots)$$

Collect like powers and use (3)-(4) to arrive at $O(1)$ and $O(\varepsilon)$ problems.

$O(1)$ problem

(5) $$y_0'' + y_0 = 0$$

(6) $$y_0(0) = 0 \qquad y_0'(0) = 1$$

has the solution

$$y_0(\tau) = \sin \tau$$

$O(\varepsilon)$ problem

(7) $$y_1'' + y_1 = y_0 (y_0'')^2 - 2\omega_1 y_0''$$

(8) $$y_1(0) = 0 \qquad y_1'(0) = -\omega_1$$

Given $y_0(\tau)$ above the differential equation (7) $\Rightarrow$

$$y_1'' + y_1 = \sin^3\tau + 2\omega_1 \sin\tau$$

TRIG IDENT

$$y_1'' + y_1 = \underbrace{(2\omega_1 + \tfrac{3}{4})}_{\text{must vanish}} \sin\tau - \frac{1}{4}\sin 3\tau$$

To eliminate secular terms

$$\boxed{\omega_1 = -\frac{3}{8}}$$

yields the $O(\varepsilon)$ problem

(7′) $$y_1'' + y_1 = -\frac{1}{4}\sin 3\tau$$

(8′) $$y_1(0) = 0 \qquad y_1'(0) = +\frac{3}{8}$$

whose soln (after some work) is

$$y_1(\tau) = \frac{9}{32}\sin\tau + \frac{1}{32}\sin(3\tau)$$

General Solution

$$y(t,\varepsilon) = \sin\tau + \frac{\varepsilon}{32}(9\sin\tau + \sin 3\tau) + O(\varepsilon^2)$$

$$\tau = \left(1 - \frac{3}{8}\varepsilon + O(\varepsilon^2)\right)t$$

Period Correction

$$T(\varepsilon) = \frac{2\pi}{\omega(\varepsilon)}$$

$$T(\varepsilon) = 2\pi\left(1 - \frac{3}{8}\varepsilon + \cdots\right)^{-1}$$

$$T(\varepsilon) = 2\pi\left(1 + \frac{3}{8}\varepsilon + O(\varepsilon^2)\right)$$

perturbation increases period.

QUESTION TWO

a) when $n > 1$ we can use L'Hopital:

$$\lim_{\varepsilon \to 0} \frac{\ln(1+\varepsilon^n)}{\varepsilon^{n-1}} = \lim_{\varepsilon \to 0^+} \frac{n\varepsilon^{n-1}}{(1+\varepsilon^n)(n-1)\varepsilon^{n-2}}$$

$$= \lim_{\varepsilon \to 0^+} \frac{n\varepsilon}{(n-1)(1+\varepsilon^n)}$$

$$= 0$$

Hence $\ln(1+\varepsilon^n) \ll \varepsilon^{n-1}$ for $n > 1$. For $n \in (0,1]$ the limit also clearly vanishes.

b) To show $f(\varepsilon) = \varepsilon^{\varepsilon} = O(1)$ we note

$$\lim_{\varepsilon \to 0^+} \ln f(\varepsilon) = \lim_{\varepsilon \to 0^+} \varepsilon \ln \varepsilon = 0 = L$$

Thus $f(\varepsilon) \to e^L = \underline{1}$ and $\varepsilon^{\varepsilon} = O(1)$.

c) Consider

$$f(\varepsilon) = \frac{\sqrt{1-\cos\varepsilon}}{\tan\varepsilon} \frac{(\sqrt{1+\cos\varepsilon})}{(\sqrt{1+\cos\varepsilon})} = \frac{\sin\varepsilon}{\tan\varepsilon\sqrt{1+\cos\varepsilon}}$$

$$f(\varepsilon) = \frac{\cos\varepsilon}{\sqrt{1+\cos\varepsilon}}$$

Then

$$\lim_{\varepsilon \to 0^+} f(\varepsilon) = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sqrt{1-\cos\varepsilon} = O(\tan\varepsilon)$$

QUESTION THREE

Expand $f(\varepsilon)$ using Binomial Thm

(1) $\quad f(\varepsilon) = 2\left(1-\frac{\varepsilon}{2}\right)^{1/2} = 2 - \frac{1}{2}\varepsilon - \frac{1}{16}\varepsilon^2 + O(\varepsilon^3)$

Now expand $a_k \phi_k(\varepsilon)$ for $k = 0, 1, 2$. Don't need $\phi_3(\varepsilon)$ since $\phi_k = O(\varepsilon^3)$ $\forall k \geq 3$.

$$a_0 \phi_0(\varepsilon) = a_0$$

$$a_1 \phi_1(\varepsilon) = a_1 \left(\varepsilon - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)\right)$$

$$a_2 \phi_2(\varepsilon) = a_2 \left(\varepsilon^2 + O(\varepsilon^3)\right)$$

Thus

(2) $\quad a_0\phi_0 + a_1\phi_1 + a_2\phi_2 = a_0 + a_1\varepsilon + (a_2 - \tfrac{1}{2}a_1)\varepsilon^2 + O(\varepsilon^3)$

Coefficients of (1) and (2) must match $\Rightarrow$

$$a_0 = 2 \qquad a_1 = -\frac{1}{2} \qquad a_2 = -\frac{5}{16}$$

Conclude

$$\sqrt{4-2\varepsilon} = 2 - \frac{1}{2}\log(1+\varepsilon) - \frac{5}{16}\log(1+\varepsilon)^2 + O(\varepsilon^3)$$

QUESTION 4a) Regular root

$$x(\varepsilon) = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

To expand $\sqrt{1+x}$ note first

$$F(\varepsilon) = \sqrt{1+x(\varepsilon)} = F(0) + F'(0)\,\varepsilon + O(\varepsilon^2)$$

where $x(0) = x_0$ and $x'(0) = x_1$.

$$F'(\varepsilon) = \frac{1}{2\sqrt{1+x(\varepsilon)}}\,x'(\varepsilon)$$

$$F'(0) = \frac{1}{2\sqrt{1+x_0}}\,x_1$$

Since $x(\varepsilon)^4 = x_0^4 + O(\varepsilon)$ we arrive at the $O(1)$ and $O(\varepsilon)$ problems

$$O(1) \qquad -(1+x_0)^{1/2} + 2 = 0$$

$$O(\varepsilon) \qquad x_0^4 - \frac{1}{2\sqrt{1+x_0}}\,x_1 = 0$$

Solve sequentially: $x_0 = 3$, $x_1 = 324$

$$x(\varepsilon) = 3 + 324\,\varepsilon + O(\varepsilon^2)$$

Binomial expansion for $(1+x)^{1/2}$ would be

$$(1+x_0+\varepsilon x_1+\cdots)^{1/2} = (1+x_0)^{1/2}\left(1+\frac{\varepsilon x_1}{(1+x_0)} + O(\varepsilon^2)\right)^{1/2}$$

$$= (1+x_0)^{1/2}\left(1+\frac{\varepsilon x_1}{2(1+x_0)} + O(\varepsilon^2)\right)$$

$$= (1+x_0)^{1/2} + \frac{\varepsilon x_1}{2\sqrt{1+x_0}} + O(\varepsilon^2)$$

QUESTION 4b Singular root $\quad x = \dfrac{X}{\varepsilon^{\alpha}}, \quad \alpha > 0$

$$\varepsilon^{1-4\alpha} X^4 - \sqrt{1 + X\varepsilon^{-\alpha}} + 2 = 0$$

$$(1) \qquad \underset{\uparrow}{\varepsilon^{1-4\alpha}} X^4 - \underset{\uparrow}{\varepsilon^{-\alpha/2}} \sqrt{X + \varepsilon^{\alpha}} + 2 = 0$$

Balance of indicated terms yields

$$\alpha = 2/7$$

then (after mult. (1) thru by $\varepsilon^{\alpha/2}$) eqn (1) $\Rightarrow$

$$(2) \qquad X^4 + \sqrt{X + \delta^2} + 2\delta = 0$$

where $\delta(\varepsilon) = \varepsilon^{1/7}$. Using $X = X_0 + \delta X_1 + O(\delta^2)$

$$X_0^4 - \sqrt{X_0} = 0 \qquad O(1)$$

$$4X_0^3 X_1 - \frac{1}{2\sqrt{X_0}} X_1 + 2 = 0 \qquad O(\delta)$$

Solve sequentially: $X_0 = 1$, $X_1 = -4/7 \Rightarrow$

$$x = \frac{1}{\varepsilon^{2/7}} \left(1 - \frac{4}{7}\varepsilon^{1/7} + O(\varepsilon^{2/7}) \right)$$

Detail on expansion of $F(\delta) \equiv \sqrt{X + \delta^2} = F(0) + F'(0)\delta + \cdots$

$$F'(\delta) = \frac{1}{2\sqrt{X + \delta^2}} (X'(\delta) + 2\delta)$$

$$F'(0) = \frac{1}{2\sqrt{X_0}} X_1 \qquad F(0) = \sqrt{X_0}$$

hence

$$\sqrt{X + \delta^2} = \sqrt{X_0} + \frac{1}{2\sqrt{X_0}} X_1 \delta + O(\delta^2)$$