Math 450 (2011) – Homework 5

Due: November 21, 2011.

1. [6 pts] Let $y(x,\epsilon)$ be the solution of the nonlinear boundary value problem:

$$\begin{aligned} \epsilon y'' + (1+2x)y' - 2y &= 0 \quad , \quad 0 < x < 1 \\ y(0) &= A \quad , \quad y(1) = 3 \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to x and $0 < \epsilon \ll 1$.

a) Show that when $A = 1$ the leading order outer solution $y_0(x)$ satisfies both boundary conditions. This is an example of a case when there is no layer.

b) For the case $A = 2$, find the uniformly valid solution $y_u(x,\epsilon)$ having a boundary layer at $x = 0$. Sketch your solution.

Error Function: The next problem involves the "error function" z=erf(x) whose definition you can look up. The solution of

$$z'' + \alpha^2 x z' = 0$$

is

$$z = c_1 + c_2 \; erf\left(\frac{\sqrt{2}\alpha x}{2}\right)$$

Also, note $erf(0) = 0$ and $erf(x) \to 1$ as $x \to \infty$.

2. [6 pts] Interior Layer Example:

$$\begin{aligned} \epsilon y'' + xy' + 3x^3 y &= 0 \quad , \quad -1 < x < 1 \\ y(-1) &= 4e \quad , \quad y(1) = 2e^{-1} \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to x and $0 < \epsilon \ll 1$.

a) Let $y_\pm(x)$ be outer solutions that satisfy the boundary conditions

$$y_-(-1) = 4e \qquad , \qquad y_+(1) = 2e^{-1}$$

b) Derive and find the <u>general solution</u> of the inner layer problem for $Y_0(X)$ where $X = \frac{x}{\delta}$.

c) Your inner solution should involve two constants c_1 and c_2. Find these using the following matching conditions and then sketch $y_u(x,\epsilon)$

$$\lim_{x\to 0^-} y_-(x) = \lim_{X\to -\infty} Y_0(X) \qquad , \qquad \lim_{x\to 0^+} y_+(x) = \lim_{X\to +\infty} Y_0(X)$$

3. [6 pts] Easy Nonlinear BVP:

$$\begin{aligned} \epsilon y'' + y' - e^y &= 0 \quad , \quad 0 < x < 1 \\ y(0) &= 0 \quad , \quad y(1) = 0 \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to x and $0 < \epsilon \ll 1$. Find the uniformly valid approximation $y_u(x, \epsilon)$ having a layer at $x = 0$. In the layer, $e^y = O(1)$ so is not involved in the dominant balance. When done, sketch your solution.

4. [2 pts] Boundary Layer Thickness: Consider the differential equation

$$\epsilon x^m y'' + x^n y' + q(x) y = 0 \qquad , \qquad m > n > 0$$

If this equation has a bounded layer at $x = 0$, what is the order of the thickness? Specifically, for

$$y(x, \epsilon) = Y(X, \epsilon) \qquad , \qquad X = \frac{x}{\delta(\epsilon)} \qquad , \qquad \delta = \epsilon^p$$

there is a dominant balance of the first two terms only for some power $p > 0$ that depends on the constants m, n. What is p?