Homework 6

QUESTION ONE a)

i) $A = \{ y \in C[0,\pi] : y'(0) = 1, y'(\pi) = -1 \}$ hence set of admissible variations is

$$A^* = \{ h \in C[0,\pi] : h'(0) = h'(\pi) = 0 \}$$

Then $y + h \in A$ if $y \in A$ and $h \in A^*$.

ii)

$$F(\varepsilon) = J(y + \varepsilon h) = (y(\alpha) + \varepsilon h(\alpha))(y'(\alpha) + \varepsilon h'(\alpha))$$

Taking a derivative in ε and then $\varepsilon = 0 \Rightarrow$

$$F'(0) = \delta J(y,h) = y(\alpha)h'(\alpha) + y'(\alpha)h(\alpha)$$

whish is the product rule.

iii) Note $\bar{y}(x) = \sin x \in A$ and $h(x) = \cos x \in A^*$

$$\delta J(\bar{y}, h) = (\sin\alpha)(-\sin\alpha) + (\cos\alpha)(\cos\alpha)$$

$$\delta J(\bar{y}, h) = \cos^2\alpha - \sin^2\alpha$$

$$\delta J(\bar{y}, h) = \cos(2\alpha)$$

QUESTION ONE b)

i) $\mathcal{A} = C^2[0,1] \Rightarrow \mathcal{A}^* = C^2[0,1]$

ii)

$$F(\varepsilon) = J(y+\varepsilon h) = \int_0^1 \underbrace{\sqrt{x^2 + 2(y+\varepsilon h)(y'+\varepsilon h') + 1}}_{G(\varepsilon)}\, dx$$

Noting

$$G'(\varepsilon) = \frac{(yh'+hy') + O(\varepsilon)}{G(\varepsilon)} \rightarrow \frac{yh'+hy'}{G(0)}$$

Evaluation of $F'(0)$ then yields

$$\delta J(y,h) = \int_0^1 \frac{yh'+hy'}{\sqrt{x^2+2yy'+1}}\, dx$$

iii) For $\bar{y}(x) = x$ and $h(x) = \frac{1}{2}x + 2$

$$\delta J(\bar{y},h) = \int_0^1 \frac{x+2}{x+1}\, dx = 1 + \ln 2$$

QUESTION TWO $\mathcal{A} = \{ y \in C^2[0,1] : y(0)=0, y(1)=0$

Lagrangian for problem is

$$L(y, y') = y + \ln(1+y')$$

Euler Lagrange eqns must be satisfied:

$$L_y = \frac{d}{dx} L_{y'}$$

$$1 = \frac{d}{dx}\left(\frac{1}{1+y'}\right)$$

$$x + c_1 = (1+y')^{-1}$$

Hence the general solution is

(1) $$y(x) = \ln(x+c_1) - x + c_2$$

Both B.C. must be satisfied:

$$y(0) = \ln(c_1) + c_2 = 0$$

$$y(1) = \ln(1+c_1) - 1 + c_2 = 0$$

Subtracting equations $\Rightarrow \ln\left(\frac{c_1}{1+c_1}\right) + 1 = 0 \Rightarrow c_1 = \frac{1}{e-1}$

(2) $$c_1 = \frac{1}{e-1} \qquad c_2 = \ln(e-1)$$

Ultimately (1)-(2) give

$$y(x) = \ln\big((e-1)x+1\big) - x$$

QUESTION THREE

$$L(y, y') = 4y'^2 + 2yy' - y^2$$

$$\mathcal{A} = \{ y \in C^2[0,\pi] : y'(0) = A, \ y(\pi) = 0 \}$$

The Euler Lagrange equation is (still)

$$L_y = \frac{d}{dx} L_{y'}$$

For given Lagrangian is equivalent to

$$(1) \qquad y'' + \frac{1}{4} y = 0$$

whose general solution is

$$y(x) = c_1 \cos\left(\tfrac{1}{2}x\right) + c_2 \sin\left(\tfrac{1}{2}x\right)$$

In both cases $A = 0$, $A = 1$ must have $c_2 = 0$.

$$(2) \qquad y(x) = c_1 \cos\left(\tfrac{1}{2}x\right)$$

CASE $A = 1$ Extrema must have form (1). $\Rightarrow$

$$y'(x) = -\tfrac{1}{2} c_1 \sin\left(\tfrac{1}{2}x\right)$$

$$y'(0) = 0 \neq A = 1 \quad \text{for any } c_1$$

Conclude there is no extrema.

CASE $A = 0$

$$y'(0) = 0 \qquad \forall c_1 \in \mathbb{R}$$

Hence one parameter family of solns

$$y(x) = c_1 \cos\left(\tfrac{1}{2}x\right) \qquad \text{not unique.}$$

QUESTION FOUR

$$L(x, y') = \cos^2 x \cdot y'^2$$

$$\mathcal{A} = \{ y \in C^2[0, \tfrac{\pi}{4}] : y(0) = 1,\ y(\tfrac{\pi}{4}) = 2 \}$$

Since $L_y = 0$, $L_{y'}$ is a first integral of the Euler Lagrange equations.

$$L_{y'} = 2y'(x)\cos^2 x = c_1 \qquad c_1 \in \mathbb{R}$$

Thus

$$y' = c_1 \sec^2 x$$

$$y = c_1 \tan x + c_2$$

is the general solution.

Use boundary conditions

$$y(0) = c_2 = 1$$

$$y(\tfrac{\pi}{4}) = c_1 + c_2 = 2$$

Solution of these is unique:

$$c_1 = 1 \qquad c_2 = 1$$

Hence the sole extrema is

$$y(x) = \tan x + 1$$