Math 451 (2012) – Homework 7

Due: Friday, January 27, 2012. NAME: ____________________

1. [10 pts] Find all the natural boundary conditions associated with extremizing $J(y)$ over $\mathcal{A}$ for the following functionals and admissible sets.

(a)

$$\begin{aligned} J(y) &= y(1)^3 + \int_0^1 \left(y^2 - (x+1)^3 y'^2 \right) \, dx \\ \mathcal{A} &= \{ y : \; y \in C^2[0,1], y(0) = 1 \} \end{aligned}$$

(b)

$$\begin{aligned} J(y) &= y'(0) + \int_0^1 \left(y y'' + x y' \right) \, dx \\ \mathcal{A} &= \{ y : \; y \in C^4[0,1], y(0) = 1 \} \end{aligned}$$

Do not find the extrema. Just derive all the natural boundary conditions. It is easiest if one derives the conditions using a general Lagrangian. For example, for part (b) start with

$$J(y) = y'(0) + \int_0^1 L(x, y, y', y'') \, dx$$

for an arbitrary Lagrangian L.

2. [5 pts] Find the extrema of

$$J(y) \equiv \int_0^1 \left(\frac{1}{2} y'^2 + y' y + y' + y \right) \, dx$$

over

$$\mathcal{A} = \{ y : \; y \in C^2[0,1], \; y(0) = \frac{1}{2} \}$$

3. [5 pts] Find the extrema of

$$J(y) \equiv \int_0^1 \left(y y' + (y'')^2 \right) \, dx$$

over

$$\mathcal{A} = \{ y : \; y \in C^4[0,1], \; y(0) = 0, y'(0) = 1, y(1) = 2, y'(1) = 4 \}$$

4. [6 pts] Find the extrema of

$$J(y) \equiv \int_0^1 xy(x)\ dx$$

over

$$\mathcal{A} = \left\{ y \in C^2[0,1] : y(0) = 0, y(1) = 0 \right\}$$

subject to the constraint

$$K(y) \equiv \int_0^1 y'(x)^2\ dx = 1$$

5. [4 pts] Consider the motion of a particle in the (x, y)-plane with Lagrangian

$$L = T - V = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - V(r)$$

where $V(r)$ is the potential energy and T is the kinetic energy. Re-express the Lagrangian in polar coordinates (r, θ), i.e., $L = L(r, \dot{r}, \theta, \dot{\theta})$. Then, write out the Euler-Lagrange equations. These are the equations of motion for planar motion of a particle under the influence of a radially symmetric force. Don't solve them.

6. [5 pts] Let $\Gamma = (X(t), Y(t), Z(t))$ with $0 < t < 1$ be a geodesic on the the graph $z = y - 2x^2$. Assume that $y = y(x)$ is a function of x on Γ. Under this assumption show that for an appropriate Lagrangian $L(x, y, y')$ the length functional

$$J(y) = \int_0^1 \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}\ dt = \int_a^b L(x, y, y')\ dx$$

where $y' \equiv \frac{dy}{dx}$. If you've done things correctly, $L_y = 0$ so that

$$0 = \frac{d}{dx} L_{y'}$$

If you expand the right side out and rationalize, the resulting numerator must vanish. This results in a linear differential equation for $u(x) = y'(x)$. By solving it you may determine $u(x)$ hence $y'(x)$. In this manner, find the geodesic with

$$y(0) = 0 \qquad , \qquad y(1) = 1$$