Math 451 (2012) – Homework 7

Due: Friday, January 27, 2012. NAME: ________________________

1. [10 pts] Find all the natural boundary conditions associated with extremizing $J(y)$ over $\mathcal{A}$ for the following functionals and admissible sets.

(a)

$$\begin{aligned} J(y) &= y(1)^3 + \int_0^1 \left(y^2 - (x+1)^3 y'^2\right)\, dx \\ \mathcal{A} &= \{y:\ y \in C^2[0,1],\ y(0) = 1\} \end{aligned}$$

(b)

$$\begin{aligned} J(y) &= y'(0) + \int_0^1 (yy'' + xy')\, dx \\ \mathcal{A} &= \{y:\ y \in C^4[0,1],\ y(0) = 1\} \end{aligned}$$

Do not find the extrema. Just derive all the natural boundary conditions. It is easiest if one derives the conditions using a general Lagrangian. For example, for part (b) start with

$$J(y) = y'(0) + \int_0^1 L(x, y, y', y'')\, dx$$

for an arbitrary Lagrangian L.

2. [5 pts] Find the extrema of

$$J(y) \equiv \int_0^1 \left(\frac{1}{2}y'^2 + y'y + y' + y\right)\, dx$$

over

$$\mathcal{A} = \{y:\ y \in C^2[0,1],\ y(0) = \frac{1}{2}\}$$

3. [5 pts] Find the extrema of

$$J(y) \equiv \int_0^1 \left(yy' + (y'')^2\right)\, dx$$

over

$$\mathcal{A} = \{y:\ y \in C^4[0,1],\ y(0) = 0, y'(0) = 1, y(1) = 2, y'(1) = 4\}$$

4. [6 pts] Find the extrema of

$$J(y) \equiv \int_0^1 xy(x)\, dx$$

over

$$\mathcal{A} = \left\{ y \in C^2[0,1] : y(0) = 0, y(1) = 0 \right\}$$

subject to the constraint

$$K(y) \equiv \int_0^1 y'(x)^2\, dx = 1$$

5. [4 pts] Consider the motion of a particle in the (x, y)-plane with Lagrangian

$$L = T - V = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - V(r)$$

where $V(r)$ is the potential energy and T is the kinetic energy. Re-express the Lagrangian in polar coordinates (r, θ), i.e., $L = L(r, \dot{r}, \theta, \dot{\theta})$. Then, write out the Euler-Lagrange equations. These are the equations of motion for planar motion of a particle under the influence of a radially symmetric force. Don't solve them.

6. [5 pts] Let $\Gamma = (X(t), Y(t), Z(t))$ with $0 < t < 1$ be a geodesic on the the graph $z = y - 2x^2$. Assume that $y = y(x)$ is a function of x on Γ. Under this assumption show that for an appropriate Lagrangian $L(x, y, y')$ the length functional

$$J(y) = \int_0^1 \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}\, dt = \int_a^b L(x, y, y')\, dx$$

where $y' \equiv \frac{dy}{dx}$. If you've done things correctly, $L_y = 0$ so that

$$0 = \frac{d}{dx} L_{y'}$$

If you expand the right side out and rationalize, the resulting numerator must vanish. This results in a linear differential equation for $u(x) = y'(x)$. By solving it you may determine $u(x)$ hence $y'(x)$. In this manner, find the geodesic with

$$y(0) = 0 \quad , \quad y(1) = 1$$

QUESTION ONE a)

Admissible variations

$$\mathcal{A}^* = \{ h \in C^2[0,1] : \; h(0) = 0 \}$$

Need NBC at $x = 1$. Define $y = \bar{y} + \varepsilon h$

$$F(\varepsilon) = J(y) = (\bar{y}(1) + \varepsilon h(1))^3 + \int_0^1 L(x, y, y')\,dx$$

Find first variation

$$F'(0) = \delta J(\bar{y}, h) = 3\bar{y}(1)^2 h(1) + \int_0^1 (L_y h + L_{y'} h')\,dx$$

Integrate by parts

$$F'(0) = 3\bar{y}(1)^2 h(1) + L_{y'} h \Big|_0^1 + \int_0^1 \left(L_y - \frac{d}{dx} L_{y'}\right) h\,dx$$

Note $h(0) = 0$ but $h(1)$ need not be. Hence

$$F'(0) = \underbrace{\left(3\bar{y}(1)^2 + L_{y'}\Big|_{x=1}\right)}_{=0 \text{ NBC.}} h(1) + \int_0^1 \underbrace{\left(L_y - \frac{d}{dx} L_{y'}\right)}_{=0 \text{ ELeqn}} h\,dx$$

For $L = (y^2 - (x+1)^3 y'^2)$ have $L_{y'} = -2(x+1)^3 y'$
so NBC is

$$\boxed{3\bar{y}(1)^2 - 16\bar{y}'(1) = 0}$$

QUESTION ONE (b)

For the given A, variations have $h(0) = 0$ but otherwise $h(1), h'(0), h'(1)$ are arbitrary. For $y \equiv \bar{y} + \varepsilon h$ we have

$$F(\varepsilon) = (\bar{y}'(0) + \varepsilon h'(0)) + \int_0^1 L(x, y, y', y'')\,dx$$

Compute $F'(0) = \delta J(\bar{y}, h)$ first variation.

$$F'(0) = h'(0) + \int_0^1 (L_y h + L_{y'} h' + L_{y''} h'')\,dx$$

Integrate by parts and collect boundary terms. The $L_{y''} h''$ term must be integrated by parts twice.

$$F'(0) = \underset{\uparrow}{h'(0)} + (L_{y'} h + L_{y''} h')\Big|_{x=0}^{x=1} - \frac{d}{dx} L_{y''} h\Big|_{x=0}^{x=1} + \int_0^1 (\text{/////})\, h\,dx$$

where the EL eqn $(\text{/////}) = L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''}$. Since it must vanish and since $h(0) = 0$ we have

$$F'(0) = \overset{A}{\left(L_{y'} - \frac{d}{dx} L_{y''}\right) h\Big|_{x=1}} + \overset{B}{(L_{y''}) h'\Big|_{x=1}} + \overset{C}{(\underset{\uparrow}{1} - L_{y''}) h'\Big|_{x=0}}$$

Given $L = y y'' + x y'$ the terms A–C must vanish

A: $x - y'\big|_{x=1} = 0$ $\qquad y'(1) = 1$ } NBC

B: $y\big|_{x=1} = 0$ $\qquad y(1) = 0$ } NBC

C: $(1 - y)\big|_{x=0} = 0$ $\qquad y(0) = 1$ } NBC $=$ given.

QUESTION TWO

$$J(y) = \int_0^1 L(x, y, y')\,dx \qquad y(0) = \tfrac{1}{2}$$

where $L = \frac{1}{2}(y')^2 + yy' + y' + y$. The Natural Boundary Cond.

$$L_{y'}\Big|_{x=1} = y'(1) + y(1) + 1 \quad = 0$$

Euler Lagrange equations and Boundary Cond.
$L_y = \frac{d}{dx} L_{y'} \Rightarrow$

$$y'' = 1 \qquad \text{Euler Lagrange}$$

$$y(0) = \tfrac{1}{2} \qquad \text{Given B.C.}$$

$$y'(1) + y(1) = -1 \qquad \text{Natural B.C.}$$

General solution

$$y(x) = \tfrac{1}{2}x^2 + c_1 x + c_2$$

Two B.C. equivalent to $c_2 = \frac{1}{2}$ and

$$2 + 2c_1 = -1$$

Hence $c_1 = -\frac{3}{2}$ and

$$y(x) = \tfrac{1}{2}x^2 - \tfrac{3}{2}x + \tfrac{1}{2}$$

QUESTION THREE

$$J(y) = \int_0^1 L(x,y,y',y'')\,dx = \int_0^1 \left(yy' + (y'')^2\right) dx$$

where $y(0)=0$, $y'(0)=1$, $y(1)=2$, $y'(1)=4$.

Euler Lagrange Eqn

$$L_y - \frac{d}{dx}L_{y'} + \frac{d^2}{dx^2}L_{y''} = 0$$

becomes

$$y^{(4)}(x) = 0$$

whose general solution is

$$y(x) = c_1x^3 + c_2x^2 + c_3x + c_4$$

Four boundary conditions for c_1, c_2, c_3, c_4:

$$\begin{array}{rcrcrcrcll}
 & & & & & & c_4 & = & 0 & \quad y(0)=0 \\
 & & & & c_3 & & & = & 1 & \quad y'(0)=1 \\
c_1 & + & c_2 & + & c_3 & + & c_4 & = & 2 & \quad y(1)=2 \\
3c_1 & + & 2c_2 & + & c_3 & & & = & 4 & \quad y'(1)=4
\end{array}$$

whose solution is $c_1=1$, $c_2=0$, $c_3=1$, $c_4=0$.

$$y(x) = x^3 + x$$

QUESTION FOUR

$$J(y) = \int_0^1 L(x,y,y')\,dx \qquad L = xy$$

$$K(y) = \int_0^1 G(x,y,y')\,dx \qquad G = (y')^2$$

Admissible $y \in \mathcal{A}$ satisfy $y(0)=0$, $y(1)=0$.
The augmented Lagrangian

$$L^* \equiv L - \lambda G = xy - \lambda (y')^2$$

must satisfy $L^*_y = \frac{d}{dx} L^*_{y'}$. Thus extrema satisfy

$$(1) \qquad \left(L_y - \frac{d}{dx} \cancel{L_{y'}}\right) = \lambda\left(\cancel{G_y} - \frac{d}{dx} G_{y'}\right)$$

Explicitly

$$(2) \qquad x = -2\lambda y'' \qquad \text{EL-eqn}$$

$$(3) \qquad y(0) = y(1) = 0 \qquad \text{B.C.}$$

$$(4) \qquad \int_0^1 (y')^2\,dx = 1 \qquad \text{Integral constraint.}$$

The general solution of (2) is

$$y(x) = -\frac{1}{12\lambda} x^3 + c_1 x + c_2$$

Using the boundary conditions (3) we get

$$y(x) = \frac{1}{12\lambda}(x - x^3)$$

Determine λ from the integral constraint using

$$y'(x) = \frac{1}{12\lambda}(1-3x^2)$$

Hence

$$K(y) = \int_0^1 \frac{1}{144\lambda^2}(1-3x^2)^2\,dx$$

$$K(y) = \frac{1}{180\lambda^2}$$

The requirement $K(y) = 1$ yields

$$\lambda = \pm\frac{\sqrt{5}}{30}$$

There are two extrema:

$$y(x) = \pm\frac{\sqrt{5}}{2}(x-x^3)$$

QUESTION FIVE

For polar coordinates, $x = r\cos\theta$, $y = r\sin\theta \Rightarrow$

$$\dot{x}^2 = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2$$

$$\dot{y}^2 = (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2$$

Expanding and summing we find

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(r)$$

$$(1) \quad L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

The Euler-Lagrange Equations ar

$$L_r = \frac{d}{dt} L_{\dot{r}}$$

$$L_\theta = \frac{d}{dt} L_{\dot{\theta}}$$

For L given by (1) these are (explicitly)

$$m\ddot{r} = mr\dot{\theta}^2 - V'(r)$$

$$mr^2\dot{\theta} = c$$

where c is constent. The last equation represents conservation of angular momentum.

QUESTION SIX

Geodesic parametrization has

$$Y(t) = y(X(t))$$

$$Z(t) = y(X(t)) - 2X(t)^2$$

Thus

$$\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 = \dot{X}^2 + y'(X)^2\dot{X}^2 + (y'(X) - 4X)^2\dot{X}^2$$

and

$$(1) \qquad J(X, Y) = \int_0^1 \dot{X}\sqrt{1 + y'(X)^2 + (y'(X) - 4X)^2}\, dt$$

This is an integral in t. Use the substitution

$$x \equiv X(t) \qquad dx = \dot{X}\, dt$$

to rewrite (1) as

$$J(y) = \int_a^b \sqrt{1 + y'^2 + (y' - 4x)^2}\, dx$$

Here $a = X(0)$, $b = X(1)$. The Lagrangian is

$$L(x, y') = \sqrt{1 + y'^2 + (y' - 4x)^2}$$

Since $L_y = 0$, extrema of $J(y)$ must satisfy

$$L_{y'} = \text{constant}$$

$$L_{y'} = \frac{2y' - 4x}{\sqrt{1 + y'^2 + (y' - 4x)^2}} = c$$

Solving for $y'(x)$

$$y'(x) = 2x + \frac{1}{2}\sqrt{\frac{2c^2}{2 - c^2}}\ \sqrt{1 + 8x^2}$$

$$(2) \quad y'(x) = 2x + \lambda \sqrt{1 + 8x^2}$$

where $\lambda \in \mathbb{R}$ is some constant. Integrate (2) on $x \in (0,1)$ using given B. Cond.

$$\underbrace{y(1) - y(0)}_{1} = \underbrace{\int_0^1 2x\,dx}_{1} + \lambda \int_0^1 \sqrt{1 + 8x^2}\,dx$$

from which we conclude $\lambda = 0$ (since the integral is positive). Usin this in (2)

$$y'(x) = 2x$$

and hence

$$y(x) = x^2$$

The geodescic on $z = y - 2x^2$ thus has a parametrization

$$(X(t), Y(t), Z(t)) = (t, t^2, -t^2) \qquad t \in (0, 1)$$