Math 451 (2012) – Homework 8

(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Wednesday, Feb. 22, 2012. NAME: ______________________

1. [5 pts] Let $\phi_0(x) = 1, \phi_1(x) = \frac{1}{2} - x$. It is easily verified $< \phi_0, \phi_1 >= 0$ in $L^2[0,1]$.

a) Find constants a, b so that $\phi_2(x) = 6x^2 + ax + b$ is orthogonal to both ϕ_0 and ϕ_1:

$$\begin{aligned} < \phi_0, \phi_2 > &= 0 \\ < \phi_1, \phi_2 > &= 0 \end{aligned}$$

b) Use the orthogonality properties of ϕ_k to find the Fourier coefficients c_k in the expansion

$$f(x) = x^2 + x + 3 = \sum_{k=0}^{k=2} c_k \ \phi_k(x)$$

2. [3 pts] Let $a = y - x, b = z - y, c = z - x$ where x, y, z are elements of any normed inner product space. If $\| \cdot \|$ is the inner product induced norm, i.e., $\| a \|^2 = < a, a >$, prove

$$\| c \|^2 = \| a \|^2 + \| b \|^2$$

is true if $a \perp b$, i.e., $< a, b >= 0$. This is Pythagorus's theorem.

3. [7 pts] For every function $f(x) \in L^2[0, \pi]$ there are c_n such that

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$

a) Compute c_n for $f(x) = x$ noting $\cos n\pi = (-1)^n$.

b) Use Parseval's identity and the result in a) to determine a formula for S where

$$S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

4. [4pts] Let $Lu = -(pu')' + qu$ be a Sturm Liouville operator with either Dirichlet or Neumann boundary conditions. Show if $p(x) > 0$ and $q(x) > 0$ on $[a, b]$ the eigenvalues of L are all positive. Consider $< u, Lu >= \lambda < u, u >$ and integrate by parts.

5. [6pts] Given $\{\sin nx\}_{n \geq 1}$ is an orthogonal basis for $L^2[0, \pi]$ find the Fourier coefficients c_n in

$$f(x) = \sum_{n=1}^{\infty} c_n \sin nx$$

for $f(x) = 1, x, \cos x$.

5. [15 pts] Below are three regular Sturm-Liouville eigenvalue Problems (SLP)

$$
\begin{array}{llll}
(I) & y'' + \lambda y = 0 & y'(0) = 0 & y(1) = 0 \\
(II) & y'' + \lambda y = 0 & y(0) + 2y'(0) = 0 & 3y(2) + 2y'(2) = 0 \\
(III) & \frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{\lambda}{x}y = 0 & y(1) = 0 & y(e) = 0
\end{array}
\tag{1}
$$

Find all eigenvalues λ_n and associated eigenfunctions $y_n(x)$ for each of (I)-(III) above. When possible find an explicit formula for λ_n as in $\lambda_n = n^2$. If you can not find an explicit formula, λ_n will be roots of some function $f(z)$ as in $f(\lambda_n) = 0$. In those cases state what $f(z)$ is.

Remarks: The last problem (III) is Cauchy-Euler and is a slight generalization of what we've done in class. A more general regular SLP eigenvalue problem has the form

$$
-\frac{d}{dx}\left(x\frac{dy}{dx}\right) + q(x)y(x) = \lambda r(x)y
$$

where $r(x) > 0$. For us $r(x) \equiv 1$. All the same theorems apply if one uses the weighted inner product:

$$
< u, v >= \int_a^b r(x)u(x)v(x)\ dx
$$

In particular, eigenvalues are orthogonal under the weighted inner product but not the $L^2[a, b]$ inner product.