Math 450 (2009) – Homework 4

Due: November 6, 2009. NAME: ________________________

1. [10 pts] Let $y(t, \epsilon)$ be the solution of the initial value problem

$$\begin{aligned} y'' + y &= \epsilon\, y(y')^2 \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 0 \quad , \quad y'(0) = 1 \end{aligned}$$

where $(\;)'$ denotes differentiation with respect to t. Assume

$$\begin{aligned} y(t, \epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon) \equiv 1 + \omega_1 \epsilon + O(\epsilon^2) \end{aligned}$$

where $y_k(\tau)$ are $2\pi-$periodic in τ for appropriate choices of ω_k for $k \geq 1$. Use Poincare Lindstedt's method to find $y_0(\tau)$, $y_1(\tau)$ and the corrected period of the oscillation, i.e., T_0 and T_1 in the exact period (in the original time t):

$$T(\epsilon) = T_0 + \epsilon T_1 + O(\epsilon^2)$$

You will need to look up appropriate trigonometric identities to complete the problem.

2. [9 pts] <u>Prove</u> that as $\epsilon \to 0^+$ the following are true:

$$\begin{aligned} e^{-1/\epsilon} &\ll \epsilon^n \qquad , \quad \forall n > 0 \\ \int_0^\epsilon f(x)\, dx &= O(\epsilon) \\ log(\epsilon) &\ll \frac{1}{1 - cos(\epsilon)} \end{aligned}$$

For the first, consider the log of the ratio to make the conclusion. The middle can be proved using the Fundamental Theorem of Calculus and the last can be shown using L'Hospital's rule (though there is a simpler way).

3. [6pts] An asymptotic sequence $\{\phi_n(\epsilon)\}$ is defined by $\phi_n(\epsilon) = sin^n \epsilon$ for $n \geq 0$ noting $\phi_0 = 1$. Find constants a_0, a_1, a_2 and a_3 such that

$$f(\epsilon) \equiv \sqrt{1 - 4\epsilon} \sim a_0\phi_0(\epsilon) + a_1\phi_1(\epsilon) + a_2\phi_2(\epsilon) + a_3\phi_3(\epsilon) + O(\phi_4) \quad as \quad \epsilon \to 0$$

Hint: expand both sides in powers of ϵ.

4. [10 pts] Consider the equation

$$f(x,\epsilon) = \epsilon x^2 - \sqrt{x} + 1 = 0$$

Using calculus one can prove that there are exactly two positive roots to the above equation. If you plot ϵx^2 and $\sqrt{x} - 1$ you can quickly see that for ϵ small, one root is $O(1)$ and the other is singular in ϵ.

a) Compute x_0, x_1 in the regular expansion

$$\bar{x}_-(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

b) For the singular root, determine X_0, X_1 and α in the expansion

$$\bar{x}_+(\epsilon) = \frac{1}{\epsilon^\alpha}\left(X_0 + \delta X_1 + O(\delta^2)\right) \quad , \quad \alpha > 0$$

for an appropriate function $\delta(\epsilon) \ll 1$.

5. [10 pts] Find the leading inner and outer solutions $y_0(x)$ and $Y_0(X)$ of the boundary value problem

$$\begin{aligned} \epsilon y'' + y' + y^2 &= 0 \quad , \quad x \in (0,1) \\ y(0) &= \frac{1}{4} \quad , \\ y(1) &= \frac{1}{2} \quad , \end{aligned}$$

and then a uniformly valid approximation $y_u(x,\epsilon)$.

QUESTION ONE

Derivative expansions

$$y'(t) = y_0'(\tau) + (y_1'(\tau) + \omega_1 y_0'(\tau))\varepsilon + O(\varepsilon^2)$$

$$y''(t) = y_0''(\tau) + (y_1''(\tau) + 2\omega_1 y_0''(\tau))\varepsilon + O(\varepsilon^2)$$

Substituting into

$$y'' + y = \varepsilon y (y')^2 \qquad y(0) = 0 \qquad y'(0) = 1$$

we obtain

$O(1)$ $\quad y_0'' + y_0 = 0 \quad , \ y_0(0) = 0 \ , \ y_0'(0) = 1$

$O(\varepsilon)$ $\quad y_1'' + y_1 = y_0(y_0')^2 - 2\omega_1 y_0'' \quad , \ y_1(0) = 0 \ , \ y_1'(0) = -\omega$

The solution of the O(1) problem is

$$y_0(\tau) = \sin\tau$$

Using this in the $O(\varepsilon)$ differential eqn.

$$y_1'' + y_1 = \sin\tau\cos^2\tau + 2\omega_1 \sin\tau = g(\tau)$$

Expose the seculanity using trig identities

$$\begin{aligned} g(\tau) &= \sin\tau(1 - \sin^2\tau) + 2\omega_1 \sin\tau \\ &= (1 + 2\omega_1)\sin\tau - \underline{\sin^3\tau} \\ &= (1 + 2\omega_1)\sin\tau - \frac{3}{4}\sin\tau + \frac{1}{4}\sin 3\tau \end{aligned}$$

Thus

$$g(\tau) = \left(\frac{1}{4} + 2\omega_1\right)\sin\tau + \frac{1}{4}\sin 3\tau$$

HERE WE USED $\sin^3\tau = \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau$

In other words

$$y_1'' + y_1 = \underbrace{\left(\frac{1}{4} + 2\omega_1\right)}_{} \sin\tau \quad + \frac{1}{4}\sin 3\tau$$

must vanish
to eliminate
secular terms

$$\omega_1 = -\frac{1}{8}$$

Thus the choice $\omega_1 = -\frac{1}{8}$ yields

$$y_1'' + y_1 = \frac{1}{4}\sin 3\tau \qquad y_1(0) = 0, \; y_1'(0) = \frac{1}{8}$$

The general solution is

$$y_1(\tau) = c_1 \cos\tau + c_2 \sin\tau - \frac{1}{32}\sin(3\tau)$$

Applying initial conditions

$$y_1(\tau) = \frac{1}{32}\left(7\sin\tau - \sin(3\tau)\right)$$

Period Correction $\quad \omega = 1 - \frac{1}{8}\varepsilon + o(\varepsilon^2)$

$$T(\varepsilon) = \frac{2\pi}{\omega(\varepsilon)} = 2\pi\left(1 - \frac{1}{8}\varepsilon + o(\varepsilon^2)\right)^{-1}$$

$$T(\varepsilon) = 2\pi\left(1 + \frac{1}{8}\varepsilon + O(\varepsilon^2)\right)$$

$$T(\varepsilon) = T_0 + \varepsilon T_1 + O(\varepsilon^2)$$

So $T_0 = 2\pi$, $T_1 = \frac{\pi}{4}$.

QUESTION TWO

a) Show $e^{-1/\varepsilon} \ll \varepsilon^n$ for all $n > 0$

$$f(\varepsilon) = \varepsilon^{-n} e^{-1/\varepsilon}$$

Suffices to show $f(\varepsilon) \ll 1$.

$$L = \ln f = -n \log \varepsilon - \frac{1}{\varepsilon}$$

$$L = -\frac{1}{\varepsilon}(1 + n\varepsilon \ln \varepsilon)$$

$$L \sim -\frac{1}{\varepsilon} \qquad \text{as} \quad \varepsilon \to 0^+$$

hence $L \to -\infty$ which is only possible if $f \to 0$.

b) By the Fundamental Theorem of Calculus (FTC)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon f(x)\,dx = f(0)$$

showing

$$\int_0^\varepsilon f(x)\,dx = O(\varepsilon)$$

If you don't recognize the FTC, note if

$$F(\varepsilon) = \int_0^\varepsilon f(x)\,dx$$

$$F(\varepsilon) = F(0) + F'(0)\varepsilon + O(\varepsilon^2)$$

$$F(\varepsilon) = 0 + f(0)\varepsilon + O(\varepsilon^2)$$

$$F(\varepsilon) = f(0)\varepsilon + O(\varepsilon^2)$$

c) Show $\log \varepsilon \ll \frac{1}{1-\cos\varepsilon}$

Need to show the following ratio $\to 0$

$$f(\varepsilon) = \frac{\log \varepsilon}{(1-\cos\varepsilon)^{-1}} \qquad \frac{\infty}{\infty} \text{ form}$$

L'Hopital

$$\lim_{\varepsilon\to 0} f(\varepsilon) = \lim_{\varepsilon\to 0} \frac{\varepsilon^{-1}}{-(1-\cos\varepsilon)^{-2}(\sin\varepsilon)}$$

$$= -\lim_{\varepsilon\to 0} \frac{(1-\cos\varepsilon)^2}{\varepsilon\sin\varepsilon} \qquad \frac{0}{0} \text{ form}$$

$$= -\lim_{\varepsilon\to 0} \frac{2(1-\cos\varepsilon)\sin\varepsilon}{\sin\varepsilon + \varepsilon\cos\varepsilon}$$

$$= -\lim_{\varepsilon\to 0} \frac{2(1-\cos\varepsilon)}{1 + \underbrace{\varepsilon\cot\varepsilon}_{\text{approaches 1}}}$$

$$= 0$$

A much simpler way would be to note from a Taylor series

$$(1-\cos\varepsilon) = \varepsilon^2 F(\varepsilon) \qquad F(0) \neq 0$$

Then

$$(1-\cos\varepsilon)\log\varepsilon = \underbrace{\varepsilon^2 \log\varepsilon}_{\downarrow\ 0}\ \underbrace{F(\varepsilon)}_{\downarrow\ F(0)\neq 0} \to 0$$

Proves the result.

QUESTION THREE

Using the Binomial Theorem (after some work)

(1) $$\sqrt{1-4\varepsilon} = 1 - 2\varepsilon - 2\varepsilon^2 - 4\varepsilon^3 + O(\varepsilon^4)$$

Using the fact that

$$\sin\varepsilon = \varepsilon - \frac{1}{6}\varepsilon^3 + O(\varepsilon^5)$$

the right side can be expanded

$$\begin{aligned}
a_0\phi_0 &= a_0 \\
a_1\phi_1 &= a_1\varepsilon - \frac{1}{6}a_1\varepsilon^3 + O(\varepsilon^5) \\
a_2\phi_2 &= a_2\varepsilon^2 + O(\varepsilon^5) \\
a_3\phi_3 &= a_3\varepsilon^3 + O(\varepsilon^5
\end{aligned}$$

Thus the sum

(2) $$\sum_{k\geq 0} a_k\phi_k = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \left(a_3 - \frac{1}{6}a_1\right)\varepsilon^3 + O(\varepsilon^5$$

The coefficients of ε^n in (1)-(2) must match, so

$$a_0 = 1 \qquad a_1 = -2 \qquad a_2 = -2$$

and

$$a_3 - \frac{1}{6}a_1 = -4$$

yields

$$a_3 = -4\frac{1}{3} = -\frac{13}{3}$$

QUESTION FOUR $\boxed{\varepsilon x^2 - \sqrt{x} + 1 = 0}$

a) Find regular soln of the form

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

Substitution into $f(x,\varepsilon) = 0$ yields

$$\varepsilon(x_0 + \varepsilon x_1 + \cdot)^2 - \sqrt{x_0}\,(1 + \varepsilon \frac{x_1}{x_0} + \cdot\,\cdot)^{1/2} + 1 = 0$$

Binomial Thm on 2nd term and collect powers

$O(1)$ $$\sqrt{x_0} - 1 = 0$$

$O(\varepsilon)$ $$x_0^2 - \frac{1}{2}\frac{x_1}{\sqrt{x_0}} = 0$$

Solving yields $x_0 = 1,\ x_1 = 2 \Rightarrow$

$$\bar{x}_{-}(\varepsilon) = 1 + 2\varepsilon + O(\varepsilon^2)$$

b) To find singular root first rescale x

$$x = \frac{X}{\varepsilon^\alpha} \qquad \alpha > 0$$

where $\alpha > 0$ is to be determined. Then $f(x,\varepsilon) = 0$ becomes

$$\varepsilon^{1-2\alpha} X^2 - \varepsilon^{-\frac{1}{2}\alpha}\sqrt{X} + 1 = 0$$

(1) $$\underset{①}{X^2} - \underset{②}{\varepsilon^{\frac{3}{2}\alpha - 1}\sqrt{X}} + \underset{③}{\varepsilon^{2\alpha-1}} = 0$$

$$① \qquad ② \gg ③$$

Since ② >> ③ for all $\alpha > 0$ must balance first two terms for non trivial leading X

(2) $$\alpha = \frac{2}{3}$$

Using the choice $\alpha = 2/3$, eqn (1) becomes

(2) $$X^2 - \sqrt{X} + \delta(\varepsilon) = 0 \quad , \quad \delta = \varepsilon^{1/3}$$

Now assume expansion

(3) $$X = X_0 + \delta X_1 + O(\delta^2)$$

Using (3) in (2) and expanding in δ

$O(1)$ $$X_0^2 - \sqrt{X_0} = 0$$

$O(\delta)$ $$2 X_0 X_1 + 1 - \frac{X_1}{2\sqrt{X_0}} = 0$$

Solution of $O(1)$ problem is $X_0 = 1$ which when used in $O(\delta)$ problem yields $X_1 = -2/3$

$$\alpha = \frac{2}{3} \qquad X_0 = 1 \qquad X_1 = -\frac{2}{3} \qquad \delta = \varepsilon^{1/3}$$

Conclude

$$x \sim \frac{1}{\varepsilon^{2/3}} \left(1 - \frac{2}{3} \varepsilon^{1/3} + O(\varepsilon^{2/3}) \right)$$

$$x \sim \frac{1}{\varepsilon^{2/3}} - \frac{2}{3} \frac{1}{\varepsilon^{1/3}} + O(1)$$

QUESTION FIVE

(1) $$\varepsilon y'' + y' + y^2 = 0 \qquad x \in (0,1)$$

(2) $$y(0) = \frac{1}{4} \qquad y(1) = \frac{1}{2}$$

Outer solution (assuming BL at $x=0$)

$$y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$

yields leading order outer problem

(3) $$y_0' + y_0^2 = 0 \quad , \quad y_0(1) = \frac{1}{2}$$

whose solution is

$$y_0(x) = \frac{1}{1+x}$$

Inner solution

Let

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\varepsilon^\alpha} \qquad \alpha > 0$$

yields

$$Y'' + \varepsilon^{\alpha-1} Y' + \varepsilon^{2\alpha-1} Y^2 = 0$$

$$① \sim ② \gg ③$$

Must choose $\alpha = 1$ so that $① \sim ②$.

$$Y'' + Y' + \varepsilon Y^2 = 0$$

Assume

$$Y(X, \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$

then

$$(4) \qquad Y_0'' + Y_0' = 0 \qquad Y_0(0) = \frac{1}{4}$$

is the leading order inner problem noting Y_0 satisfies the left $x=0$ boundary condition. The solution of (4) is

$$Y_0(X) = A\,(e^{-X} - 1) + \frac{1}{4}$$

where A is a constant.

Matching

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

$$M = 1 = \frac{1}{4} - A$$

so that $A = -3/4$ and

$$Y_0(X) = 1 - \frac{3}{4} e^{-X}$$

Uniform solution

$$y_u(x, \varepsilon) = Y_0\left(\frac{x}{\varepsilon}\right) + y_0(x) - M$$

$$y_u(x, \varepsilon) = \frac{1}{(1+x)} - \frac{3}{4} e^{-x/\varepsilon}$$