Math 450 (2009) – Homework 5

Due: November 20, 2009. NAME: ____________________

1. [12 pts] Let $y(x,\epsilon)$ be the solution of the nonlinear boundary value problem:

$$\begin{aligned} \epsilon y'' + x(y')^2 - y &= 0 \quad , \quad 0 < x < 1 \\ y(0) &= A \quad , \quad y(1) = B \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to x and $0 <\ll 1$.

a) Let $A = -1$ and $B = 1$. Find all uniformly valid approximations $y_u(x,\epsilon)$ that have a boundary layer at $x = 0$. There are several (two or more) such solutions since the outer problem has several solutions, i.e., the solution is not unique.

b) Sketch your solutions.

This problem is meant to illustrate that nonlinear problems may have non-unique solutions.

2. [8 pts] Find a uniformly valid approximation to the (singular) initial value problem:

$$\begin{aligned} \epsilon y'' + (t+1)^2 y' &= 1 \quad , \quad t > 0 \\ y(0) &= 1 \quad , \quad \epsilon y'(0) = 1 \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to t and $0 < \epsilon \ll 1$.

QUESTION ONE

(1) $$\varepsilon y'' + x(y')^2 - y = 0$$

(2) $$y(0) = -1 \qquad y(1) = 1$$

Outer problem $y(x,\varepsilon) = y_0(x) + o(1)$

(3) $$x(y_0')^2 - y_0 = 0 \qquad y_0(1) = 1$$

is separable

$$\left(\frac{dy_0}{dx}\right)^2 = \frac{y_0}{x}$$

$$\frac{dy_0}{dx} = \pm\sqrt{\frac{y_0}{x}}$$

whose general solutions are

$$y_0^{\pm}(x) = (c \pm \sqrt{x})^2 \qquad c \in \mathbb{R}$$

Try to satisfy boundary condition

$$y_0^{+}(1) = (c+1)^2 = 1 \quad \Rightarrow \quad c = 0, -2$$

$$y_0^{-}(1) = (c-1)^2 = 1 \quad \Rightarrow \quad c = 0, 2$$

Hence we have

$$\left.\begin{array}{ll} y_0^{+}(x) = x & y_0^{+}(x) = (\sqrt{x} - 2)^2 \\ y_0^{-}(x) = x & y_0^{-}(x) = (-\sqrt{x} + 2)^2 \end{array}\right\} \text{same}$$

or only two different outer solns

(4) $$y_0(x) = x \qquad y_0(x) = (2 - \sqrt{x})^2$$

Inner problem $\quad y(x,\varepsilon) = Y(X,\varepsilon)\ , \quad X = \dfrac{x}{\delta(\varepsilon)}$

Substitution, noting $x = \delta X$, yields

$$\frac{\varepsilon}{\delta^2} Y'' + (\delta X)\left(\frac{1}{\delta} Y'\right)^2 - Y = 0$$

$$(5) \qquad \frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} X (Y')^2 - Y = 0$$

$$①\ \sim\ ②\ \gg\ ③$$

Since ② $\gg$ ③ for all $\delta \ll 1$, choose δ s.t. ① $\sim$ ② or

$$\delta(\varepsilon) = \varepsilon \qquad \text{B-Layer thickness}$$

Then (5) becomes

$$Y'' + X Y'^2 - \varepsilon Y = 0$$

For $Y(X,\varepsilon) = Y_0(X) + o(1)$ we find the leading order inner problem

$$Y_0'' + X (Y_0')^2 = 0 \qquad Y_0(0) = -1$$

This is separable. Let $u = Y_0'(X)$ then

$$u' + x u^2 = 0$$

Solving we find

$$u(X) = Y_0'(X) = \frac{1}{\frac{1}{2}X^2 + c^2} \qquad c \in \mathbb{R}$$

Note "$-c^2$" not possible else $Y_0'(X)$ undefined as $X \to \infty$.

Integrating in X yields the general solution

$$Y_0(X) = \frac{\sqrt{2}}{c} \arctan\left(\frac{X}{\sqrt{2}\,c}\right) + d$$

where $c, d \in \mathbb{R}$ are unknown constants. The boundary condition $y(0) = -1$ implies

$$Y_0(0) = d = -1$$

So that $Y_0(X)$ is known up to the constant c

$$(6) \qquad Y_0(X) = \frac{\sqrt{2}}{c} \arctan\left(\frac{X}{\sqrt{2}\,c}\right) - 1$$

<u>Uniform solution using $y_0(x) = x$ outer</u>

Matching

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

$$M = 0 = \frac{\sqrt{2}}{c}\,\frac{\pi}{2} - 1$$

Hence $c = \pi/\sqrt{2}$ and

$$y_u(x, \varepsilon) = y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - M$$

$$y_u(x, \varepsilon) = x + \frac{2}{\pi} \arctan\left(\frac{x}{\pi \varepsilon}\right) - 1$$

Uniform solution using $y_0(x) = (2-\sqrt{x})^2$

Matching

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

$$M = 4 = \frac{\pi}{\sqrt{2}\, C} - 1$$

hence $c = \pi/(5\sqrt{2})$

Here the inner solution is slightly different

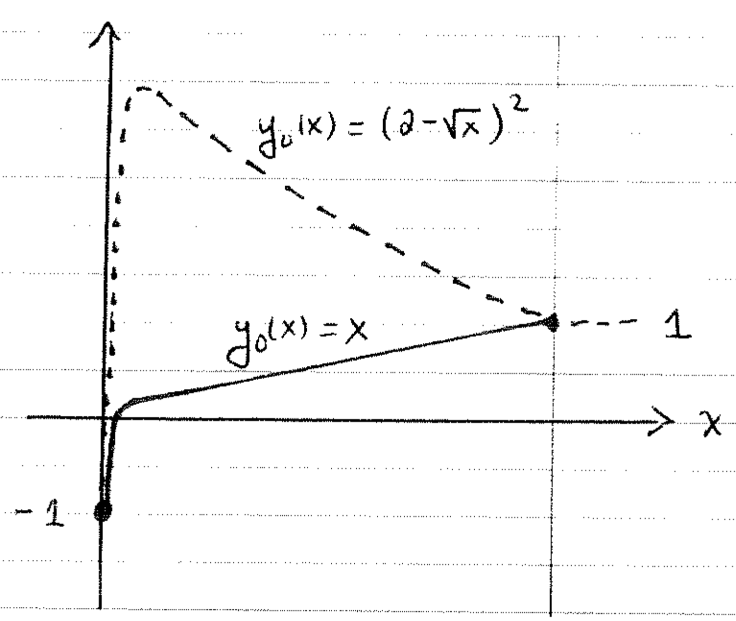
$$Y_0(X) = \frac{10}{\pi} \arctan\left(\frac{5X}{\pi}\right) - 1$$

and

$$y_u(x,\varepsilon) = y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - M$$

$$y_u(x,\varepsilon) = (2-\sqrt{x})^2 + \frac{10}{\pi} \arctan\left(\frac{5x}{\pi\varepsilon}\right) - 5$$

Graphs of both



QUESTION TWO

$$\varepsilon y'' + (t+1)^2 y' = 1 \quad , \quad y(0) = 1 \quad , \quad \varepsilon y'(0) = 1$$

outer problem (satisfies no I.C.)

$$(t+1)^2 y_0' = 1$$

$$y_0 = C - \frac{1}{(t+1)}$$

where C is to be found from matching.

inner problem $\quad y(t,\varepsilon) = Y(\tau,\varepsilon)$, $\tau = \dfrac{t}{\delta(\varepsilon)}$

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{(1+\delta\tau)^2 Y'}{\delta} = 1 \quad , \quad \frac{\varepsilon}{\delta} Y'(0) = 1$$

$$① \sim ②$$

Also want $O(1)$

For the choice ① ~ ②, $\delta(\varepsilon) = \varepsilon$ and one gets

$$Y'' + (1+\varepsilon\tau)^2 Y' = \varepsilon \quad , \quad Y'(0) = 1$$

For $Y = Y_0(\tau) + o(1)$ the $O(1)$ inner problem is

$$Y_0'' + Y_0' = 0 \qquad Y_0(0) = 1 \quad , \quad Y_0'(0) = 1$$

Yields the solution

$$Y_0(\tau) = 2 - e^{-\tau}$$

Matching

$$M = \lim_{t \to 0^+} y_0(t) = \lim_{\tau \to \infty} Y_0(\tau)$$

$$M = c - 1 = 2 \qquad \Rightarrow c = 3$$

so that the completed outer soln is

$$y_0(t) = 3 - \frac{1}{(t+1)} \qquad (M = 2)$$

Uniform Soln

$$y_u(t, \varepsilon) = y_0(t) + Y_0\left(\frac{t}{\varepsilon}\right) - M$$

$$y_u(t, \varepsilon) = 3 - \frac{1}{(t+1)} - e^{-t/\varepsilon}$$