Math 450 (2009) – Homework 6

Due: December 4, 2009. NAME: ____________________

1. [8 pts] For each of the following functionals compute $J(\bar{y})$ and the first variation $\delta J(\bar{y}, h)$.

a) Let $J : \mathcal{A} \to \mathbb{R}$ where $\mathcal{A} = C[0,2]$

$$J(y) \equiv y(1)^3 \quad , \quad \bar{y}(x) = x(2-x) \quad , \quad h(x) = 3x$$

a) Let $J : \mathcal{A} \to \mathbb{R}$ where $\mathcal{A} = C^2[0,1]$

$$J(y) \equiv \int_0^1 \left(x^2 - y(x)^2 + y'(x)^2\right)\, dx \quad , \quad \bar{y}(x) = x \quad , \quad h(x) = x^2$$

2. [4 pts] Use the Euler Lagrange equations to find the extrema $\bar{y}(x)$ of

$$J(y) \equiv \int_0^{\pi/6} (sec^2 x)\; y'(x)^3\; dx$$

over the admissible set

$$\mathcal{A} = \{y :\; y \in C^2[0, \pi/6],\; y(0) = 1,\; y(\pi/6) = 3/2\}$$

3. [4 pts] Use the Euler Lagrange equations to find <u>all</u> possible extrema $\bar{y} \in \mathcal{A}$ of

$$\begin{aligned} J(y) \;&\equiv\; \int_0^{\pi} y'(x)^2 - \; y(x)^2\; dx \\ \mathcal{A} \;&=\; \{y :\; y \in C^2[0, \pi],\; y(0) = 0,\; y(\pi) = 0\} \end{aligned}$$

4. [6 pts] The functional $J(y)$ is defined by

$$\begin{aligned} J(y) \;&\equiv\; \int_0^{\pi/2} \sqrt{y(x)^2 + y'(x)^2}\; dx \\ \mathcal{A} \;&=\; \{y :\; y \in C^2[0, \pi/2],\; y(0) = 1,\; y(\pi/2) = 1/2\;\} \end{aligned}$$

a) The Euler-Lagrange equation for this problem are nonlinear and messy. Nevertheless, show that the equation can be simplified to:

$$y^2 + 2(y')^2 - yy'' = 0$$

b) Use the substitution $y(x) = u(x)^{-1}$ to find the extrema $\bar{y} \in \mathcal{A}$ for J.

5. [8 pts] A functional $J : \mathcal{A} \to \mathrm{I\!R}$ is defined by

$$J(y) \equiv \int_0^1 \frac{1}{1 + y(x)^2}\, dx$$

where the admissible set

$$\mathcal{A} \equiv \{ y : \ y(x) = \alpha(x-1) \ for \ some \ \alpha \in \mathrm{I\!R} \ \}$$

This is just the set of linear functions through $(x, y) = (1, 0)$.

a) Let

$$F(\alpha) \equiv J(y_\alpha) \qquad where \qquad y_\alpha(x) = \alpha(x-1)$$

$F(\alpha)$ is a real valued function. Using a graphing calculator, software or calculus, plot F as a function of α.

b) Given your graph, what $\bar{y} \in \mathcal{A}$ maximizes J over $\mathcal{A}$?

c) Given your graph, what is the range $R(J)$ of the functional J?

d) Does J attain its minimum value, i.e., is there a $\bar{y} \in \mathcal{A}$ such that

$$J(\bar{y}) = \min_{y \in \mathcal{A}} J(y)$$

e) Compute the norms $\| \ y_\alpha \ \|_\infty$ and $\| \ y_\alpha \ \|_1$