Math 450 (2009) – Homework 6

Due: December 4, 2009. NAME: ____________________

1. [8 pts] For each of the following functionals compute $J(\bar{y})$ and the first variation $\delta J(\bar{y}, h)$.

a) Let $J : \mathcal{A} \to \mathrm{IR}$ where $\mathcal{A} = C[0, 2]$

$$J(y) \equiv y(1)^3 \quad , \quad \bar{y}(x) = x(2-x) \quad , \quad h(x) = 3x$$

a) Let $J : \mathcal{A} \to \mathrm{IR}$ where $\mathcal{A} = C^2[0, 1]$

$$J(y) \equiv \int_0^1 \left(x^2 - y(x)^2 + y'(x)^2\right) \, dx \quad , \quad \bar{y}(x) = x \quad , \quad h(x) = x^2$$

2. [4 pts] Use the Euler Lagrange equations to find the extrema $\bar{y}(x)$ of

$$J(y) \equiv \int_0^{\pi/6} (sec^2 x) \; y'(x)^3 \, dx$$

over the admissible set

$$\mathcal{A} = \{y : \; y \in C^2[0, \pi/6], \; y(0) = 1, \; y(\pi/6) = 3/2\}$$

3. [4 pts] Use the Euler Lagrange equations to find <u>all</u> possible extrema $\bar{y} \in \mathcal{A}$ of

$$\begin{aligned} J(y) &\equiv \int_0^{\pi} y'(x)^2 - y(x)^2 \, dx \\ \mathcal{A} &= \{y : \; y \in C^2[0, 1], \; y(0) = 0, \; y(\pi) = 0\} \end{aligned}$$

4. [6 pts] The functional $J(y)$ is defined by

$$\begin{aligned} J(y) &\equiv \int_0^{\pi/2} \sqrt{y(x)^2 + y'(x)^2} \, dx \\ \mathcal{A} &= \{y : \; y \in C^2[0, 1], \; y(0) = 1, \; y(\pi/2) = 1/2 \; \} \end{aligned}$$

a) The Euler-Lagrange equation for this problem are nonlinear and messy. Nevertheless, show that the equation can be simplified to:

$$y^2 + 2(y')^2 - yy'' = 0$$

b) Use the substitution $y(x) = u(x)^{-1}$ to find the extrema $\bar{y} \in \mathcal{A}$ for J.

5. [8 pts] A functional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$J(y) \equiv \int_0^1 \frac{1}{1 + y(x)^2}\, dx$$

where the admissible set

$$\mathcal{A} \equiv \{ y : \ y(x) = \alpha(x-1) \ \mathit{for\ some}\ \alpha \in \mathbb{R} \ \}$$

This is just the set of linear functions through $(x, y) = (1, 0)$.

a) Let

$$F(\alpha) \equiv J(y_\alpha) \qquad \mathit{where} \qquad y_\alpha(x) = \alpha(x-1)$$

$F(\alpha)$ is a real valued function. Using a graphing calculator, software or calculus, plot F as a function of α.

b) Given your graph, what $\bar{y} \in \mathcal{A}$ maximizes J over $\mathcal{A}$?

c) Given your graph, what is the range $R(J)$ of the functional J?

d) Does J attain its minimum value, i.e., is there a $\bar{y} \in \mathcal{A}$ such that

$$J(\bar{y}) = \min_{y \in \mathcal{A}} J(y)$$

e) Compute the norms $\| \ y_\alpha \ \|_\infty$ and $\| \ y_\alpha \ \|_1$

QUESTION ONE (a) $\quad J(y) = y(1)^3$

$$J(y+\varepsilon h) = F(\varepsilon) = (y(1)+\varepsilon h(1))^3$$

$$F'(\varepsilon) = 3(y(1)+\varepsilon h(1))^2 \cdot h(1)$$

$$F'(0) = 3\, y(1)^2 h(1)$$

Thus for any $y \in A$, $h \in A$ we have

$$\delta J(y,h) = 3\, y(1)^2 h(1)$$

For $\bar{y}(x) = x(2-x)$ and $h(x) = 3x$ $\quad J(\bar{y}) = 1$

$$\delta J(\bar{y},h) = 3(1)^2(3) = 9$$

QUESTION ONE (b) $\quad J(y) = \int_0^1 (x^2 - y(x)^2 + y'(x)^2)\,dx$

For $\bar{y}(x) = x$, $h(x) = x^2$ we have

$$J(\bar{y}+\varepsilon h) = \int_0^1 x^2 - (x+\varepsilon x^2)^2 + (1+2\varepsilon x)^2\,dx$$

After some calculations

$$F(\varepsilon) = J(\bar{y}+\varepsilon h)$$

$$F(\varepsilon) = 1 + \tfrac{3}{2}\varepsilon + \tfrac{17}{15}\varepsilon^2$$

$$F'(0) = \tfrac{3}{2}$$

$$\delta J(\bar{y},h) = \tfrac{3}{2} \qquad J(\bar{y}) = 1$$

QUESTION TWO

$$J(y) = \int_0^{\pi/6} \sec^2 x \; y'(x)^3 \, dx$$

where the Lagrangian $L(x, y, y') = \sec^2 x \, (y')^2$

$$L_y = \frac{d}{dx} L_{y'}$$

$$0 = \frac{d}{dx} \underbrace{\left(3 \sec^2 x \; y'(x)^2 \right)}_{\text{must be constant}}$$

Hence an extremal of J satisfies

$$\cancel{3} \sec^2 x \; y'(x)^2 = \cancel{3} A^2 \qquad A \in \mathbb{R}$$

$$y'(x) = A \cos x$$

$$(1) \qquad y(x) = A \sin x + B$$

Eqn (1) is the general soln of the EL-eqns. For $y \in \mathcal{A}$ it must also satisfy the boundary conditions

$$y(0) = B = 1$$

$$y\left(\tfrac{\pi}{6}\right) = A \sin\left(\tfrac{\pi}{6}\right) + B = \tfrac{3}{2}$$

Hence $B = 1$, $A = 1$ and

$$\bar{y}(x) = \sin x + 1$$

is the sole extremal.

QUESTION THREE

$$J(y) = \int_0^{\pi} y'(x)^2 - y(x)^2 \, dx$$

where Lagrangian $L(x, y, y') = (y')^2 - y^2$.

$$L_y = \frac{d}{dx} L_{y'}$$

$$-2y = \frac{d}{dx}(2y')$$

Hence $\bar{y}$ must satisfy

$$(1) \qquad y'' + y = 0 \qquad y(0) = 0 \quad y(\pi) = 0$$

The solution of (1) is not unique

$$\bar{y}(x) = A \sin x$$

solves (1) for all $A \in \mathbb{R}$. Thus there are many extrema.

In this case one can show

$$J(\bar{y}) = 0 \qquad \forall A \in \mathbb{R}$$

i.e, there are many extrema and J has the same value at each.

QUESTION FOUR (a)

$$L = (y^2 + y'^2)^{1/2} \qquad \text{Lagrangian}$$

One way to solve the Euler Lagrange Eqns

(1) $$L_y = \frac{d}{dx} L_{y'}$$

in this case is to note that a first integral is

(2) $$L - y' L_{y'} = c$$

since L does not depend on x explicitly.

Instead here we will expand (1) out directly

$$L_y = \frac{y}{(y^2+y'^2)^{1/2}} = \frac{y}{L} = yL^{-1}$$

$$L_{y'} = \frac{y'}{(y^2+y'^2)^{1/2}} = \frac{y'}{L} = y'L^{-1}$$

Thus the EL-eqn is

$$yL^{-1} = \frac{d}{dx}(y'L^{-1})$$

$$= y''L^{-1} + y'\frac{d}{dx}L^{-1}$$

$$= y''L^{-1} - y'L^{-2}\frac{dL}{dx}$$

$$= y''L^{-1} - y'L^{-2}(L_y y' + L_{y'} y'')$$

Again using $L_y = yL^{-1}$, $L_{y'} = y'L^{-1}$ we have

$$yL^{-1} = y''L^{-1} - y'L^{-2}(yy'L^{-1} + y'y''L^{-2})$$

multiply by L^3

$$yL^2 = y''L^2 - y'(yy' + y'y'')$$

$$y(y^2+y'^2) = y''(y^2 + \cancel{y'^2}) - y'(yy' + y'\cancel{y''})$$

Hence

$$y^3 + 2yy'^2 - y^2y'' = 0$$

and upon dividing by $y(x) > 0$

$$(3) \quad \boxed{y^2 + 2y'^2 - yy'' = 0}$$

This is the Euler Lagrange Eqn. It is 2nd order and nonlinear.

QUESTION FOUR (b)

To solve (3) we let

$$y(x) = u(x)^{-1}$$
$$y'(x) = -u(x)^{-2}u'(x)$$
$$y''(x) = +2u(x)^{-3}u'(x)^2 - u(x)^{-2}u''(x)$$

Substitute these into eqn (3).

$$(u^{-1})^2 + 2(-u^{-2}u')^2 - u^{-1}(2u^{-3}u'^2 - u^{-2}u'') = 0$$

$$u^{-2} + \cancel{2u^{-4}u'^2} - \cancel{2u^{-4}u'^2} + u^{-3}u'' = 0$$

Multiply thru by u^3 to get

$$u'' + u = 0$$

whose general soln is

$$u(x) = A\cos x + B\sin x \qquad A, B \in \mathbb{R}$$

and hence

$$\bar{y}(x) = \frac{1}{A\cos x + B\sin x}$$

is the general soln of EL eqn.

Since extremal $\bar{y} \in \mathcal{A}$ have $y(0) = 1$, $y(\frac{\pi}{2}) = \frac{1}{2}$

$$\bar{y}(0) = \frac{1}{A} = 1$$

$$\bar{y}(\tfrac{\pi}{2}) = \frac{1}{B} = \frac{1}{2}$$

yields $A = 1$, $B = 2$ and

$$\boxed{\bar{y}(x) = \frac{1}{\cos x + 2\sin x}}$$

QUESTION FIVE

$$J(y) \equiv \int_0^1 \frac{1}{1+y(x)^2}\,dx$$

$$A = \{ y : y(x) = \alpha(x-1),\ \alpha \in \mathbb{R} \}$$

Part a)

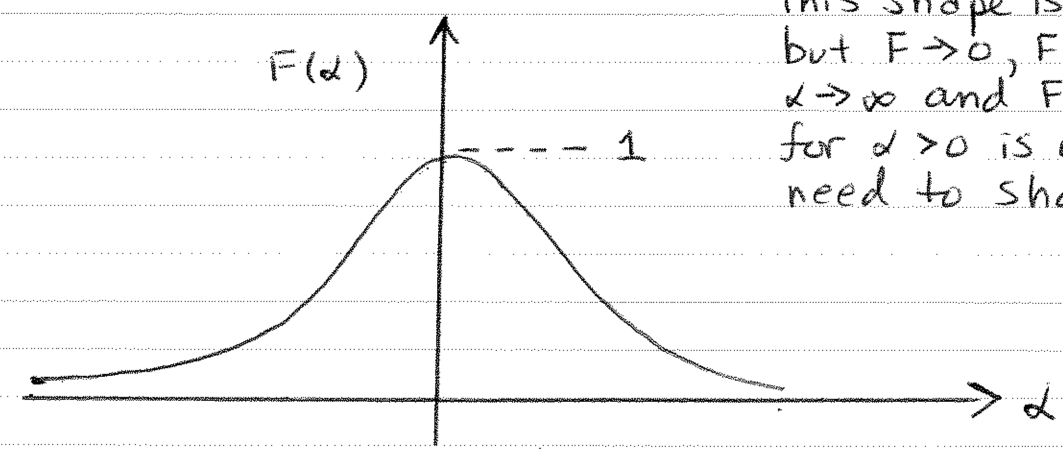
$$F(\alpha) = \int_0^1 \frac{1}{1+\alpha^2(x-1)^2}\,dx$$

$$F(\alpha) = \frac{1}{\alpha}\arctan\alpha \qquad \alpha \neq 0$$

When $\alpha = 0$, $J(0) = 1$ is well defined and equal to

$$\lim_{\alpha \to 0} \frac{1}{\alpha}\arctan\alpha = 1$$

by l'Hopital's rule.



Using calculus to prove this shape is VERY hard but $F \to 0$, $F > 0$ as $\alpha \to \infty$ and $F'(\alpha) < 0$ for $\alpha > 0$ is what you need to show.

Part b) $\bar{y}(x) \equiv 0 \qquad \alpha = 0$

maximizes $J(y)$ over $\mathcal{A}$

Part c) Since $F(\alpha) \in (0, 1]$ we have

$$R(J) = (0, 1]$$

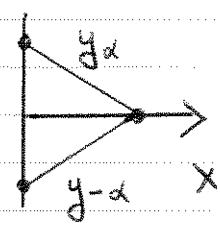
as the range of $J : \mathcal{A} \to \mathbb{R}$.

Part d) While there are y_α such that $J(y_\alpha) \to 0$ the value 0 is not attained by any $\bar{y} \in \mathcal{A}$, i.e.

$\nexists\, \bar{y} \in \mathcal{A}$ such that

$$J(\bar{y}) = \lim_{\alpha \to \infty} J(y_\alpha) = 0$$

Part e) Norm computations



$$\| y_\alpha \|_\infty = \max_{x \in [0,1]} |\alpha (x-1)| = |\alpha|$$

$$\| y_\alpha \|_1 = \int_0^1 |\alpha(x-1)|\, dx$$

$$= |\alpha| \int_0^1 |x-1|\, dx$$

$$= |\alpha| \int_0^1 (1-x)\, dx = |\alpha| \left(x - \tfrac{1}{2} x^2 \right) \Big|_0^1$$

$$= \tfrac{1}{2} |\alpha|$$