Math 450 (2009) – Homework 7

Due: Wed. January 27, 2010. NAME: ______________________

1. [7 pts] Find the extrema of

$$J(y) \equiv \int_0^1 y''(x)^2 + 240\ xy(x)\ dx$$

over the admissible set

$$\mathcal{A} = \{y :\ y \in C^4[0,1],\ y(0) = y(1) = y'(0) = y'(1) = 0\}$$

2. [7 pts] Find the extrema of

$$J(y) \equiv \int_1^e \frac{1}{2}x^2 y'(x)^2 - \frac{1}{8}y(x)^2\ dx$$

over

$$\mathcal{A} = \{y :\ y \in C^2[1,e],\ y(1) = 1\}$$

Here the Euler-Lagrange equations are of a "Cauchy-Euler" type.

3. [7 pts] Find all the natural boundary conditions associated with extremizing

$$J(y) \equiv \frac{1}{4}y'(0)^4 + \int_0^1 y(x) + \frac{1}{2}y(x)'^2 + \frac{1}{3}y''(x)^3\ dx$$

over

$$\mathcal{A} = \{y :\ y \in C^4[0,1],\ y(0) = 3, y'(1) = 2\}$$

Do not attempt to solve the Euler-Lagrange equations!

4. [7 pts] Find the extrema of

$$J(y) \equiv \int_0^1 \frac{1}{12}\ y'(x)^2\ dx$$

over

$$\mathcal{A} = \left\{y \in C^2[0,1] : y(0) = 0, y(1) = 1\right\}$$

subject to the constraint

$$K(y) \equiv \int_0^1 xy(x)\ dx = 1$$

5. [7 pts] A geodesic Γ between points P and Q on surface S is that curve (on S) connecting points P and Q having the shortest length. Great circles are geodesics on spheres (paths airplanes take to minimize distance travelled for instance). In this problem we examine the geodesics on a cone.

In cartesian coordinates the equation of a cone is $x^2 + y^2 = a^2 z^2$ for some constant a. Thus, a parametrization of a curve on the cone is given by

$$\begin{aligned} \mathbf{X}(t) &= (x(t), y(t), z(t)) \\ \mathbf{X}(t) &= (r(t)\cos\theta(t), r(t)\sin\theta(t), a\ r(t)) \end{aligned}$$

Without loss of generality we scale t so that $P = \mathbf{X}(0)$ and $Q = \mathbf{X}(1)$. Thus, the length of such a curve is

$$J(\mathbf{X}) = \int_0^1 \parallel \dot{\mathbf{X}}(t) \parallel \ dt = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ dt$$

where $(\dot{\ })$ means differentiation with respect to t.

a) Show that the Lagrangian L in

$$J(r,\theta) = \int_0^1 \parallel \dot{\mathbf{X}}(t) \parallel \ dt = \int_0^1 \ L(\theta, \dot{\theta}, r, \dot{r})\ dt$$

is, for $b^2 = a^2 + 1$,

$$L = \sqrt{b^2\ \dot{r}^2 + r^2\ \dot{\theta}^2}$$

b) Since L depends on two independent functions $r(t)$ and $\theta(t)$ there are two coupled Euler-Lagrange equations defining the geodesics on the cone. Use one of the two Euler-Lagrange equations to show that for some constant k_1

$$\frac{d\theta}{dt} = \frac{k_1 b}{r\sqrt{r^2 - k_1^2}}\ \frac{dr}{dt}$$

c) Assuming r is a function of θ, use the substitution $r = k_1 \sec u$ to show that

$$r = R(\theta) \equiv k_1 \sec\left(b^{-1}\theta + k_2\right)$$

on geodesics for constants k_1, k_2 (determined from endpoint constraints). The geodesic then has the parametrization:

$$\mathbf{X}(\theta) = (R(\theta)\cos\theta, R(\theta)\sin\theta, a\ R(\theta))$$