Math 450 (2009) – Homework 7

Due: Wed. January 27, 2010. NAME: Solns.

1. [7 pts] Find the extrema of

$$J(y) \equiv \int_0^1 y''(x)^2 + 240\, xy(x)\, dx$$

over the admissible set

$$\mathcal{A} = \{y : \ y \in C^4[0,1],\ y(0) = y(1) = y'(0) = y'(1) = 0\}$$

2. [7 pts] Find the extrema of

$$J(y) \equiv \int_1^e \frac{1}{2}x^2 y'(x)^2 - \frac{1}{8}y(x)^2\, dx$$

over

$$\mathcal{A} = \{y : \ y \in C^2[1,e],\ y(1) = 1\}$$

Here the Euler-Lagrange equations are of a "Cauchy-Euler" type.

3. [7 pts] Find all the natural boundary conditions associated with extremizing

$$J(y) \equiv \frac{1}{4}y'(0)^4 + \int_0^1 y(x) + \frac{1}{2}y(x)'^2 + \frac{1}{3}y''(x)^3\, dx$$

over

$$\mathcal{A} = \{y : \ y \in C^4[0,1],\ y(0) = 3, y'(1) = 2\}$$

Do not attempt to solve the Euler-Lagrange equations!

4. [7 pts] Find the extrema of

$$J(y) \equiv \int_0^1 \frac{1}{12}\, y'(x)^2\, dx$$

over

$$\mathcal{A} = \left\{y \in C^2[0,1] : y(0) = 0, y(1) = 1\right\}$$

subject to the constraint

$$K(y) \equiv \int_0^1 xy(x)\, dx = 1$$

5. [7 pts] A geodesic Γ between points P and Q on surface S is that curve (on S) connecting points P and Q having the shortest length. Great circles are geodesics on spheres (paths airplanes take to minimize distance travelled for instance). In this problem we examine the geodesics on a cone.

In cartesian coordinates the equation of a cone is $x^2 + y^2 = a^2 z^2$ for some constant a. Thus, a parametrization of a curve on the cone is given by

$$\begin{aligned}\mathbf{X}(t) &= (x(t), y(t), z(t)) \\ \mathbf{X}(t) &= (r(t)\cos\theta(t), r(t)\sin\theta(t), a\, r(t))\end{aligned}$$

Without loss of generality we scale t so that $P = \mathbf{X}(0)$ and $Q = \mathbf{X}(1)$. Thus, the length of such a curve is

$$J(\mathbf{X}) = \int_0^1 \| \dot{\mathbf{X}}(t) \| \; dt = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

where $(\dot{\ })$ means differentiation with respect to t.

a) Show that the Lagrangian L in

$$J(r,\theta) = \int_0^1 \| \dot{\mathbf{X}}(t) \| \; dt = \int_0^1 L(\theta, \dot{\theta}, r, \dot{r})\, dt$$

is, for $b^2 = a^2 + 1$,

$$L = \sqrt{b^2\, \dot{r}^2 + r^2\, \dot{\theta}^2}$$

b) Since L depends on two independent functions $r(t)$ and $\theta(t)$ there are two coupled Euler-Lagrange equations defining the geodesics on the cone. Use one of the two Euler-Lagrange equations to show that for some constant k_1

$$\frac{d\theta}{dt} = \frac{k_1 b}{r\sqrt{r^2 - k_1^2}} \frac{dr}{dt}$$

c) Assuming r is a function of θ, use the substitution $r = k_1 \sec u$ to show that

$$r = R(\theta) \equiv k_1 \sec\left(b^{-1}\theta + k_2\right)$$

on geodesics for constants k_1, k_2 (determined from endpoint constraints). The geodesic then has the parametrization:

$$\mathbf{X}(\theta) = (R(\theta)\cos\theta, R(\theta)\sin\theta, a\; R(\theta))$$

QUESTION ONE

$$J(y) = \int_0^1 L(x, y, y', y'')\,dx$$

where $L = (y'')^2 + 240xy$ is the Lagrangian.
Euler Lagrange Eqns are:

$$L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0$$

$$240x + 0 + 2y'''' = 0$$

yields the BVP for $y \in \mathcal{A}$.

$$(1) \qquad y^{(4)}(x) = -120x$$

$$(2) \qquad y(0) = y(1) = y'(0) = y'(1) = 0$$

General solution of (1) is

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 - x^5$$

Boundary conditions yield 4 eqns for C_K as follows

$y(0) = 0$	C_1	$= 0$
$y(1) = 0$	$C_1 + C_2 + C_3 + C_4 - 1$	$= 0$
$y'(0) = 0$	C_2	$= 0$
$y'(1) = 0$	$C_2 + 2C_3 + 3C_4 - 5$	$= 0$

Solving for $C_1 = 0,\ C_2 = 0,\ C_3 = -2,\ C_4 = 3 \Rightarrow$

$$\bar{y}(x) = -2x^2 + 3x^3 - x^5$$

QUESTION TWO

$$J(y) = \int_1^e L(x, y, y')\,dx \quad , \quad L = \frac{1}{2}x^2 y'^2 - \frac{1}{8}y^2$$

Since $y(1) = 1$ is specified, need NBC at $x = e$

$$(1) \qquad L_{y'}\Big|_{x=e} = x^2 y'(x)\Big|_{x=e} = e^2 y'(e)$$

must vanish, or

$$(2) \qquad y'(e) = 0 \qquad \text{(NBC)}$$

Euler Lagrange Eqns

$$L_y = \frac{d}{dx} L_{y'}$$

$$-\frac{1}{4}y = \frac{d}{dx}(x^2 y')$$

yields

$$x^2 y'' + 2x y' + \frac{1}{4}y = 0$$

whose general solution is $\quad (y(x) = x^r \Rightarrow r = -\frac{1}{2} \text{ root})$

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 \ln x}{\sqrt{x}}$$

Using the boundary conditions

$$y(1) = 1 \qquad y'(e) = 0$$

we find $c_1 = c_2 = 1$ or

$$\bar{y}(x) = \frac{1}{\sqrt{x}} + \frac{\ln x}{\sqrt{x}}$$

QUESTION THREE

Here

$$J(y) = \frac{1}{4} y'(0)^4 + \int_0^1 L(x, y, y', y'')\, dx$$

where

$$(1) \qquad L = y + \frac{1}{2} y'^2 + \frac{1}{3} y''^3$$

and

$$A = \{ y \in C^4[0,1] : y(0) = 3, \ y'(1) = 2 \}$$

For $\delta y = y(x) + \varepsilon h(x) \in A$ must have

$$(2) \qquad h(0) = 0 \qquad h'(1) = 0$$

for admissible variations $h(x) \in A^*$.

Compute First variation $\delta J(y,h)$

$$\delta J = y'(0)^3 h'(0) + \int_0^1 (L_y h + \underbrace{L_{y'} h' + L_{y''} h''}_{\text{must integrate by parts}})\, dx$$

Integrating by parts

$$\delta J = y'(0)^3 h'(0)$$

$$+ L_{y'} h \Big|_{\cancel{0}}^{1} + L_{y''} h' \Big|_{0}^{\cancel{1}} - \left(\frac{d}{dx} L_{y''}\right) h \Big|_{\cancel{0}}^{1}$$

$$+ \int_0^1 \underbrace{\left(L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''}\right)}_{=0 \quad (\text{EL eqn})} h(x)\, dx$$

For extrema, integral term vanishes (EL eqn).
Also condition (2) implies certain boundary terms vanish, i.e.

$$h(0) = 0 \qquad h'(1) = 0$$

but $h(1)$ and $h'(0)$ need not vanish.

First variation simplifies

$$(3) \quad \delta J = \underbrace{\left(L_{y'} - \frac{d}{dx} L_{y''}\right)}_{=0} h \Big|_{x=1} + \underbrace{\left(y'^3 - L_{y''}\right)}_{=0} h' \Big|_{x=0}$$

Since $L = y + \frac{1}{2}(y')^2 + \frac{1}{3}(y'')^3$ we have

$$L_{y'} = y'$$

$$L_{y''} = (y'')^2$$

$$\frac{d}{dx} L_{y''} = 2y''y'''$$

Using in (3) above

$$(4) \quad L_{y'} - \frac{d}{dx} L_{y''} = y' - 2y''y'''$$

$$(5) \quad y'^3 - L_{y''} = y'^3 - (y'')^2$$

In summary the two NBC are:

$$\boxed{\begin{aligned} \overset{\overset{2}{\downarrow}}{y'(1)} - 2y''(1)\,y'''(1) &= 0 \\ y'(0)^3 - y''(0)^2 &= 0 \end{aligned}} \qquad \text{NBC}$$

<u>Euler Lagrange Equations</u>

$$L_y = 1 \qquad L_{y'} = y' \qquad L_{y''} = (y'')^2$$

yield

$$L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0$$

$$1 - y'' + \frac{d^2}{dx^2}(y'')^2 = 0$$

Extrema would be solutions of

$$\frac{d^2}{dx^2}(y'')^2 - y'' + 1 = 0$$

satisfying the given BC.

$$y(0) = 3 \qquad y'(1) = 2$$

and the NBC above.

QUESTION FOUR

$$J(y) = \int_0^1 \frac{1}{12} y'(x)^2 \, dx \qquad K(y) = \int_0^1 x y(x) \, dx = 1$$

Augmented Lagrangian

$$L^*(x, y, y') = \frac{1}{12} y'^2 + \lambda x y$$

Euler Lagrange eqn

$$L^*_y = \frac{d}{dx} L^*_{y'}$$

is

$$y'' = 6 \lambda x$$

whose general soln is

$$y(x) = \lambda x^3 + c_1 x + c_2$$

for constants $c_1, c_2 \in \mathbb{R}$.

Boundary Conditions

$$y(0) = \qquad\qquad c_2 = 0$$

$$y(1) = \lambda + c_1 + c_2 = 1$$

Thus $c_1 = 1 - \lambda$ and $c_2 = 0$.

$$\bar{y}(x) = \lambda x^3 + (1-\lambda) x$$

satisfies EL eqns and B.C.

Must chose λ so that $K(\bar{y}) = 1$

$$K(\bar{y}) = \int_0^1 x\,\bar{y}(x)\,dx$$

$$= \int_0^1 \lambda x^4 + (1-\lambda)x^2\,dx$$

$$= \frac{1}{5}\lambda + \frac{1}{3}(1-\lambda)$$

$$= 1$$

only if $\lambda = -5$. Thus the extrema for the isopeimetric problem is

$$\bar{y}(x) = -5x^3 + 6x$$

QUESTION FIVE

a) Finding Lagrangian

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\dot{z} = a\dot{r}$$

Using $\sin^2\theta + \cos^2\theta = 1$ then direct algebra yields

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + a^2\dot{r}^2$$

From which

$$L = \|\dot{X}\| = \sqrt{b^2\dot{r}^2 + r^2\dot{\theta}^2} \qquad b^2 = 1 + a^2$$

b) The two EL eqns are

$$L_\theta = \frac{d}{dt} L_{\dot{\theta}}$$

$$L_r = \frac{d}{dt} L_{\dot{r}}$$

Since L does not depend on θ explicitly, $L_\theta = 0$ so that $L_{\dot{\theta}}$ is a first integral,

$$(1) \qquad L_{\dot{\theta}} = \frac{r^2\dot{\theta}}{\sqrt{b^2\dot{r}^2 + r^2\dot{\theta}^2}} = k_1$$

for some constant k_1.

Equation (1) can be solved for $\dot{\theta}$, i.e.

$$r^4\dot{\theta}^2 = k_1^2(b^2\dot{r}^2 + r^2\dot{\theta}^2)$$

$$(r^4 - k_1^2 r^2)\dot{\theta}^2 = k_1^2 b^2 \dot{r}^2$$

from which

$$(2) \qquad \dot{\theta} = \frac{k_1 b}{r\sqrt{r^2 - k_1^2}}\,\dot{r}$$

c) If $r = r(\theta)$ then (2) is separable

$$\boxed{\frac{k_1\, dr}{r\sqrt{r^2 - k_1^2}} = \frac{1}{b}\, d\theta}$$

Let

$$r = k_1 \sec u$$

$$dr = k_1 \sec u \tan u$$

$$r^2 - k_1^2 = k_1^2 \tan^2 u$$

simplifies above to

$$du = \frac{1}{b}\, d\theta$$

$$u = \frac{1}{b}\theta + k_2 \qquad , \quad k_2 \in \mathbb{R}.$$

$$\sec^{-1}(r/k_1) = \frac{1}{b}\theta + k_2$$

$$r = k_1 \sec(b^{-1}\theta + k_2)$$