

Math 450 (2009) – Homework 8
(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Wednesday, Feb. 10, 2010.

NAME: _____

1. [6 pts] Define $f_\lambda(x) \equiv \sin(\lambda x) \in L^2[0, \pi]$ for $\lambda \in \mathbb{R}$. Here λ need not be an integer.

a) Compute $F(\lambda) \equiv \|f_\lambda\|$ for all $\lambda \in \mathbb{R}$ and show F is constant on the integers.

b) Compute $G(\lambda, \mu) \equiv \langle f_\lambda, f_\mu \rangle$ for $\lambda \neq \pm\mu$

c) Show that for nonzero λ, μ the functions f_λ and f_μ are orthogonal only if

$$h(\lambda) = h(\mu) \quad \text{where} \quad h(z) = \frac{\tan(\pi z)}{z}$$

Aside from getting you to compute norms and inner products the result in c) can be used to show that if one fixes μ at some value then there exists $\lambda_n \neq \mu$ such that $\{\phi_n(x)\} = \{\sin(\lambda_n x)\}_{n \geq 1}$ is an orthogonal set much like $\{\sin(nx)\}_{n \geq 1}$ was shown to be in class.

2. [6 pts] The Legendre polynomials are a sequence of n^{th} degree polynomials $\{P_n(x)\}_{n \geq 1}$ defined by:

$$P_0(x) = 1 \tag{1}$$

$$P_1(x) = x \tag{2}$$

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad n = 1, 2, \dots \tag{3}$$

a) Use the recursion formula above to write out $P_2(x), P_3(x)$ and $P_4(x)$.

b) It can be shown that $\{P_n(x)\}$ are mutually orthogonal in $L^2[-1, 1]$. Use this fact to determine a_n (for all n) in the following expansion

$$1 + x + x^2 = \sum_{n=0}^{\infty} a_n P_n(x)$$

3. [6 pts] For every function $f(x) \in L^2[0, \pi]$ there are c_n such that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

a) Compute c_n for $f(x) = \sin(\beta x)$ where β is not an integer (note 1b) above is the same)

b) Use Parseval's identity and the result in a) to determine a formula for S where

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(\beta^2 - n^2)^2}$$

4. [4pts] The sets

$$S_1 = \{\phi_n(x)\} = \{\sin(nx)\}_{n \geq 1}$$

$$S_2 = \{\psi_n(x)\} = \{\cos(nx)\}_{n \geq 0}$$

are (complete) orthogonal sets on $L^2[0, \pi]$.

a) For each set define (clearly) an (SLP) whose eigenfunctions are ϕ_n, ψ_n

b) Find c_n and d_n in the following

$$1 = \sum_{n=1}^{\infty} c_n \phi_n(x) = \sum_{n=0}^{\infty} d_n \psi_n(x)$$

5. [18 pts] Below are three regular Sturm-Liouville eigenvalue Problems (SLP)

$$(I) \quad y'' + \lambda y = 0 \quad y(0) = 0 \quad y'(1) = 0$$

$$(II) \quad y'' + \lambda y = 0 \quad y(0) + y'(0) = 0 \quad y(1) = 0 \quad (4)$$

$$(III) \quad \frac{d}{dx} \left((2+x)^2 \frac{dy}{dx} \right) + \lambda y = 0 \quad y(-1) = 0 \quad y(1) = 0$$

Find all eigenvalues λ_n and associated eigenfunctions $y_n(x)$ for each of (I)-(III) above. When possible find an explicit formula for λ_n as in $\lambda_n = n^2$. If you can not find an explicit formula, λ_n will be roots of some function $f(z)$ as in $f(\lambda_n) = 0$. In those cases state what $f(z)$ is.

Remarks: For (II) you may want to write the general solution in the form:

$$y(x) = c_1 \sin(\mu(x-1)) + c_2 \cos(\mu(x-1))$$

Also, problem (III) is Cauchy-Euler in $z = x+2$, i.e. $y(x) = (x+2)^r$ results in an equation for r in terms of λ . Take special note of the $\lambda > \frac{1}{4}$ case.