Math 450 (2009) – Homework 8

(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Wednesday, Feb. 10, 2010. NAME: ____________________

1. [6 pts] Define $f_\lambda(x) \equiv \sin(\lambda x) \in L^2[0,\pi]$ for $\lambda \in \mathbb{R}$. Here λ need not be an integer.

a) Compute $F(\lambda) \equiv \| f_\lambda \|$ for all $\lambda \in \mathbb{R}$ and show F is constant on the integers.

b) Compute $G(\lambda,\mu) \equiv < f_\lambda, f_\mu >$ for $\lambda \neq \pm\mu$

c) Show that for nonzero λ, μ the functions f_λ and f_μ are orthogonal only if

$$h(\lambda) = h(\mu) \qquad where \quad h(z) = \frac{\tan(\pi z)}{z}$$

Aside from getting you to compute norms and inner products the result in c) can be used to show that if one fixes μ at some value then there exists $\lambda_n \neq n$ such that $\{\phi_n(x)\} = \{\sin(\lambda_n x)\}_{n\geq 1}$ is an orthogonal set much like $\{\sin(nx)\}_{n\geq 1}$ was shown to be in class.

2. [6 pts] The Legendre polynomials are a sequence of n^{th} degree polynomials $\{P_n(x)\}_{n\geq 1}$ defined by:

$$\begin{aligned} P_0(x) &= 1 && (1)\\ P_1(x) &= x && (2)\\ (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x) \qquad n = 1,2,\ldots && (3) \end{aligned}$$

a) Use the recursion formula above to write out $P_2(x), P_3(x)$ and $P_4(x)$.

b) It can be shown that $\{P_n(x)\}$ are mutually orthogonal in $L^2[-1,1]$. Use this fact to determine a_n (for all n) in the following expansion

$$1 + x + x^2 = \sum_{n=1}^{\infty} a_n \; P_n(x)$$

3. [6 pts] For every function $f(x) \in L^2[0,\pi]$ there are c_n such that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

a) Compute c_n for $f(x) = sin(\beta x)$ where β is not an integer (note 1b) above is the same)

b) Use Parseval's identity and the result in a) to determine a formula for S where

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(\beta^2 - n^2)^2}$$

~~3.~~ 4. [4pts] The sets

$$S_1 = \{\phi_n(x)\} = \{\sin(nx)\}_{n\geq 1}$$
$$S_2 = \{\psi_n(x)\} = \{\cos(nx)\}_{n\geq 0}$$

are (complete) orthogonal sets on $L^2[0,\pi]$.

a) For each set define (clearly) an (SLP) whose eigenfunctions are ϕ_n, ψ_n

b) Find c_n and d_n in the following

$$1 = \sum_{n=1}^{\infty} c_n\phi_n(x) = \sum_{n=0}^{\infty} d_n\psi_n(x)$$

~~4.~~ 5. [18 pts] Below are three regular Sturm-Liouville eigenvalue Problems (SLP)

(I)	$y'' + \lambda y = 0$	$y(0) = 0$	$y'(1) = 0$
(II)	$y' + \lambda y = 0$	$y(0) + y'(0) = 0$	$y(1) = 0$
(III)	$\frac{d}{dx}\left((2+x)^2\frac{dy}{dx}\right) + \lambda y = 0$	$y(-1) = 0$	$y(1) = 0$

(4)

$y = c_1 \cos\mu(x-1) + c_2 \sin\mu(x-1)$ and no matter

Find all eigenvalues λ_n and associated eigenfunctions $y_n(x)$ for each of (I)-(III) above. When possible find an explicit formula for λ_n as in $\lambda_n = n^2$. If you can not find an explicit formula, λ_n will be roots of some function $f(z)$ as in $f(\lambda_n) = 0$. In those cases state what $f(z)$ is.

QUESTION ONE $\quad f_\lambda(x) = \sin \lambda x$

a)

$$\langle f_\lambda, f_\lambda \rangle = \int_0^\pi \sin^2 \lambda x \, dx = \frac{\pi}{2} - \frac{1}{2\lambda} \sin \pi\lambda \cos \pi\lambda$$

$$F(\lambda) = \sqrt{\langle f_\lambda, f_\lambda \rangle} = \frac{1}{\sqrt{2}} \left(\pi - \frac{1}{\lambda} \sin \pi\lambda \cos \pi\lambda \right)^{1/2}$$

b)

$$G = \int_0^\pi f_\lambda(x) \, f_\mu(x) \, dx$$

$$G = \int_0^\pi \sin \lambda x \, \sin \mu x \, dx \qquad \text{(Tables)}$$

$$G(\lambda, \mu) = \frac{\lambda \cos(\pi\lambda) \sin(\mu\pi) - \mu \sin(\pi\lambda) \cos(\pi\mu)}{\mu^2 - \lambda^2}$$

c) Vectors are orthogonal $\Leftrightarrow G \equiv 0$.
The numerator of G must vanish

$$\lambda \cos(\pi\lambda) \sin(\mu\pi) = \mu \sin(\pi\lambda) \cos(\pi\mu)$$

which can be rewritten

$$h(\mu) = h(\lambda) \qquad h(z) \equiv \frac{\tan \pi z}{z}$$

QUESTION TWO $\quad (n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

For example

$$P_2(x) = \frac{1}{2}(3xP_1(x) - P_0(x)) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

In the expansion

$$f(x) = 1 + x + x^2 = \sum_{n=0}^{\infty} a_n P_n(x)$$

$a_n = 0$ for $n \geq 3$ since $f(x)$ is degree two polynomial.
Don't need n^{th} degree $P_n(x)$ in series.

For $n = 0, 1, 2$ use

$$(1) \qquad a_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} \qquad \langle u, v \rangle = \int_{-1}^{1} u(x)v(x)\,dx$$

I used maple to compute these integrals

$$\langle f, P_0 \rangle = \frac{8}{3} \qquad \langle f, P_1 \rangle = \frac{2}{3} \qquad \langle f, P_2 \rangle = \frac{4}{15}$$

$$\langle P_0, P_0 \rangle = 2 \qquad \langle P_1, P_1 \rangle = \frac{2}{3} \qquad \langle P_2, P_2 \rangle = \frac{2}{5}$$

$$a_0 = \frac{4}{3} \qquad a_1 = 1 \qquad a_2 = \frac{2}{3}$$

Hence

$$1 + x + x_2 = \frac{4}{3}P_0(x) + P_1(x) + \frac{2}{3}P_2(x)$$

QUESTION THREE a) Given $\{\phi_n\}$ complete on $L^2[0,\pi]$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx) = \sin(\beta x)$$

where

$$c_n = \frac{\langle \sin(nx), \sin(\beta x)\rangle}{\|\sin nx\|^2}$$

For integers n, $\|\sin nx\|^2 = \pi/2$ so that

$$c_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\sin(\beta x)\,dx$$

$$c_n = \frac{2}{\pi} G(n,\beta) \qquad \text{(From 1b)}$$

$$c_n = \frac{2}{\pi}\,\frac{n\cos(n\pi)\sin(\beta\pi)}{\beta^2 - n^2}$$

$$\boxed{c_n = \frac{2}{\pi}\,\frac{n(-1)^n}{\beta^2 - n^2}\,\sin(\beta\pi)}$$

Conclude

$$\sin\beta x = f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{n(-1)^n}{\beta^2 - n^2}\sin(\beta\pi)\sin(nx)$$

QUESTION THREE b)

Let $\{\phi_n\}$ be orthogonal and $\{\hat{\phi}_n\}$ orthonormal

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) = \sum_{n=1}^{\infty} \underbrace{c_n \|\phi_n\|}_{\hat{c}_n} \hat{\phi}_n(x)$$

Parseval's identity

$$\|f\|^2 = \sum_{n=1}^{\infty} \|\hat{c}_n\|^2 = \sum_{n=1}^{\infty} \|c_n\|^2 \cdot \underbrace{\|\phi_n\|^2}_{\pi/2}$$

For us, given c_n

$$(1) \qquad \|f\|^2 = \frac{2}{\pi} \sin^2 \beta\pi \underbrace{\sum_{n=1}^{\infty} \frac{n^2}{(\beta^2 - n^2)^2}}_{S}$$

Compute $\|f\|^2$

$$\|f\|^2 = \int_0^{\pi} \sin^2(\beta x)\, dx = \frac{\pi}{2} - \frac{1}{4\beta}\sin(2\beta)$$

Using this in (1) and solving for S, $\sin 2z = 2\sin z \cos z$

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(\beta^2 - n^2)^2} = \frac{\pi^2}{4}\csc^2(\beta\pi) - \frac{\pi}{8\beta}\cot(\beta\pi)$$

QUESTION FOUR a)

$$y'' + \lambda y = 0 \qquad y(0) = y(\pi) = 0$$

generates $\phi_n(x) = \sin(nx)$, $n \geq 1$.

$$y'' + \lambda y = 0 \qquad y'(0) = y'(\pi) = 0$$

generates $\psi_n(x) = \cos(nx)$, $n \geq 0$.

QUESTION FOUR b) Careful with $n = 0$ for set S_2.

$$\|\sin nx\|^2 = \|\phi_n\|^2 = \frac{\pi}{2} \qquad n \geq 1$$

$$\|\cos nx\|^2 = \|\psi_n\|^2 = \frac{\pi}{2} \qquad n \geq 1$$

$$\|1\|^2 = \|\psi_0\|^2 = \pi \qquad n = 0$$

$$\langle 1, \sin nx \rangle = \frac{1-(-1)^n}{n} \qquad n \geq 1$$

$$\langle 1, \cos nx \rangle = 0 \qquad n \geq 1$$

Using the basis $\{\cos nx\}$ we get

$$1 = 1 = \sum_{n=0}^{\infty} d_n \psi_n(x), \quad d_n = 0 \ \forall n \geq 1$$

whereas for the basis $\{\sin nx\}$

$$1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(nx) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)x}{(2m-1)}$$

QUESTION FIVE I (No evalues $\lambda \le 0$)

Let $\lambda = \mu^2, \mu \neq 0$ in

(i) $y'' + \mu^2 y = 0 \qquad y(0) = 0 \qquad y'(1) = 0$

General solution

$$y(x) = c_1 \cos\mu x + c_2 \sin\mu x$$

Dirichlet B.C. $y(0) = c_1 = 0$ hence

$$y(x) = c_2 \sin\mu x$$

$$y'(x) = c_2\mu\cos\mu x$$

$$y'(1) = c_2\mu\cos\mu = 0$$

only at discrete

$$\mu_n = \frac{(2n-1)}{2}\pi \qquad n = 1, 2, 3$$

Thus the (nonnormalized) eigenvalue/eigenfn pairs:

$$\lambda_n = \mu_n^2 = \frac{(2n-1)^2}{4}\pi^2 \qquad n \geq 1$$

$$y_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right) \qquad n \geq 1$$

QUESTION FIVE II

$\lambda = -\mu^2 < 0$ no eigenvalues.

$$y(x) = c_1 \sinh \mu(x-1) + c_2 \cosh \mu(x-1)$$

$$y(1) = c_2 = 0$$

hence $y(x) = c_1 \sinh \mu(x-1)$. Then

$$y(0) + y'(0) = c_1 \underbrace{(\mu \cosh \mu - \sinh \mu)}_{\text{sole root at } \mu = 0} = c_1 f(\mu)$$

Since $\nexists \mu$ s.t. $f(\mu) = 0$, $\mu \neq 0$ there are no negative eigenvalues. To prove $\mu = 0$ is sole root

$$f'(\mu) = \mu \sinh \mu > \qquad \mu \neq 0$$

implies $f(\mu)$ is an increasing fn.

$\lambda = 0$ an eigenvalue

$$y(x) = c_1 x + c_2$$

Boundary conditions

$$y'(0) + y(0) = c_1 + c_2 = 0$$

$$y(1) = c_1 + c_2 = 0$$

Here $c_2 = -c_1$ hence eval/efn pair

$$\lambda_0 = 0 \qquad y_0(x) = 1 - x$$

Case $\lambda = \mu^2 > 0$ has eigenvalues

$$y(x) = c_1 \sin\mu(x-1) + c_2 \cos\mu(x-1)$$

Boundary condition at $x = 1$ is

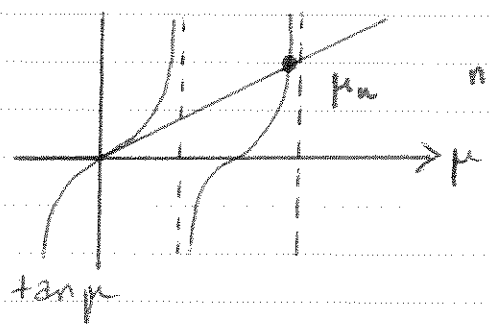
$$y(1) = c_2 = 0$$

hence $y(x) = c_1 \sin\mu(x-1)$ and

$$y(0) + y'(0) = c_1 (\mu\cos\mu - \sin\mu) = c_1 f(\mu)$$

The function $f(\mu)$ has roots μ_n since $f(\mu) = 0$ is equivalent to

$$\tan\mu = \mu$$



$n = 1, 2, \ldots$

Roots μ_n from the graphs determine eigenvalues

$$\lambda_n = {\mu_n}^2 \qquad \tan\mu_n = \mu_n$$

with associated eigenfns

$$y_n(x) = \sin\mu_n(x-1)$$

QUESTION FIVE III

Case $\lambda = \mu^2 > 0$ (eigenvalues)

(1) $$\frac{d}{dx}\left((x+2)^2 \frac{dy}{dx}\right) + \lambda y = 0$$

Is Cauchy Euler in $z = x+2$

$$z^2 y'' + 2zy' + \lambda y = 0$$

Letting $y = z^r$ yields

$$P(r) = r^2 + r + \lambda = 0$$

whose roots are

$$r_{\pm} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \lambda}$$

General soln depends on λ

(2) $y(x) = c_1 z^{r_+} + c_2 z^{r_-}$ $\qquad 0 < \lambda < \frac{1}{4}$

(3) $y(x) = c_1 z^{r_+} + c_2 z^{r_+} \ln z$ $\qquad \lambda = \frac{1}{4}$

and the complex case

(4) $y(x) = c_1 z^{\alpha} \cos(\beta \ln z) + c_2 z^{\alpha} \sin(\beta \ln z)$

when $\lambda > 1/4$ and

(5) $$r = \alpha \pm i\beta = -\tfrac{1}{2} \pm i\sqrt{\lambda - \tfrac{1}{4}}$$

we consider the latter case first noting since $z = x+2$

$$x = -1 \iff z = 1$$
$$x = +1 \iff z = 3$$

For $\lambda > \frac{1}{4}$ the boundary condition $y(-1) = 0$ is

$$y(-1) = c_1 (1)^{\alpha} \cos(0) + 0 = 0$$

hence $c_1 = 0$ and

$$y(x) = c_2 z^{\alpha} \sin(\beta \ln z)$$

Then the boundary condition $y(1) = 0 \Rightarrow$

$$y(1) = c_2 \, 3^{\alpha} \sin(\beta \ln 3) = 0$$

Hence $\beta \ln 3 = n\pi$ generates eigenvalues:

$$\beta_n \ln 3 = \sqrt{\lambda_n - \frac{1}{4}} \ln 3 = n\pi \qquad n = 1, 2, 3 \ldots$$

or

$$\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln 3}\right)^2$$

with associated eigenfunctions $(n = 1, 2, 3 \ldots)$

$$y_n(x) = (x+2)^{-1/2} \sin\left(\frac{n\pi}{\ln 3} \ln(x+2)\right)$$

For the cases $\lambda \in (0, \frac{1}{4})$ and $\lambda = \frac{1}{4}$ we use (2) - (3)

$\lambda \in (0, \frac{1}{4})$ case

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 3^{r_+} & 3^{r_-} \end{bmatrix} \end{array} \begin{array}{l} x=-1 \\ x=+1 \end{array} \qquad A\vec{c} = \vec{0}$$

clearly $\det A \neq 0$ hence $\vec{c} = \vec{0}$ and trivial solns only (no eigenvalues)

$\lambda = \frac{1}{4}$ case

$$A = \begin{bmatrix} 1 & 0 \\ 3^{r_+} & 3^{r_+} \ln 3 \end{bmatrix}$$

Again $\det A \neq 0$ hence $\lambda = \frac{1}{4}$ not an eigenvalue.

Summary

Only eigenvalues/eigenfunctions with $\lambda_n > 0$ are

$$\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln 3}\right)^2$$

$$y_n(x) = \frac{1}{\sqrt{x+2}} \sin\left(\frac{n\pi}{\ln 3} \ln(x+2)\right)$$

for $n = 1, 2, 3, \ldots$