

Taylor Series Theorems and Expansion

Taylor Series of a function of one variable: Let $f(x)$ be a smooth function of one variable x in an interval I containing a . Then the following series converges (for x near a)

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

The series is referred to as the Taylor series of $f(x)$ about $x = a$. When $a = 0$ it is called the Maclaurin series. Some common Maclaurin expansions include:

$$\begin{aligned}e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\ \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots \\ \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots, \quad |x| < 1\end{aligned}$$

Binomial Theorem An important special Taylor series is the Binomial Theorem:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

This series converges for any constant p so long as $|x| < 1$.

Taylor Series of a function of two variable: Let $f(x, y)$ be a function of two variables and suppose it has continuous partial derivatives of all orders in the neighbourhood of (x_0, y_0) . Then the following series converges:

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + H(x, y)$$

where the higher order terms H are given by

$$H(x, y) = \frac{1}{2!} (f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2) + \dots$$

Taylor Series of a vector valued function: Let $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a smooth function of $\mathbf{x} = (x, y)$ and let $\mathbf{x}_0 = (x_0, y_0)$, $\mathbf{f}(\mathbf{x}) = (f_1(x, y), f_2(x, y))$. Then

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + h.o.t.$$

where the higher order terms (*h.o.t.*) are not listed and the Jacobian matrix $D\mathbf{f}(\mathbf{x})$ is given by

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$