EXAMPLE Find the extrema of

(1) $$J(y) = \frac{1}{12}\int_0^1 (y')^2\,dx$$

over

$$A = \{ y \in C^2[0,1] : y(0) = 0, y(1) = 1 \}$$

subject to the constraint

(2) $$K(y) \equiv \int_0^1 x\, y(x)\, dx = 1$$

A necessary condition for extrema is that the augemented Lagrangian

(3) $$L^*(x,y,y') \equiv \frac{1}{12}(y')^2 + \lambda x y$$

satisfy the Euler-Lagrange Equation

(4) $$L^*_y = \frac{d}{dx} L^*_{y'}$$

For L^* defined in (3) this is, explicitly,

$$\bar{y}'' = 6\lambda x$$

whose general solution is

(5) $$\bar{y}(x) = \lambda x^3 + c_1 x + c_2$$

Goal is to find constants c_1, c_2 and λ so that $y(x)$ satisfies both boundary conditions and the integral constaint.

Boundary conditions

$$\bar{y}(0) = c_2 = 0$$

$$\bar{y}(1) = \lambda + c_1 + c_2 = 1$$

Thus

$$c_1 = 1 - \lambda \qquad c_2 = 0$$

and

$$(6) \qquad \bar{y}(x) = \lambda x^3 + (1-\lambda)x$$

Must choose λ so that $K(\bar{y}) = 1$

$$K(\bar{y}) = \int_0^1 x\,\bar{y}(x)\,dx$$

$$= \int_0^1 \lambda x^4 + (1-\lambda)x^2\,dx$$

$$= \frac{1}{5}\lambda + \frac{1}{3}(1-\lambda)$$

Require

$$K(\bar{y}) = \frac{1}{3} - \frac{2}{15}\lambda = 1$$

yields $\lambda = -5$ so that (from (6) above)

$$\boxed{\bar{y}(x) = -5x^3 + 6x}$$