Natural Boundary Conditions – Motivation

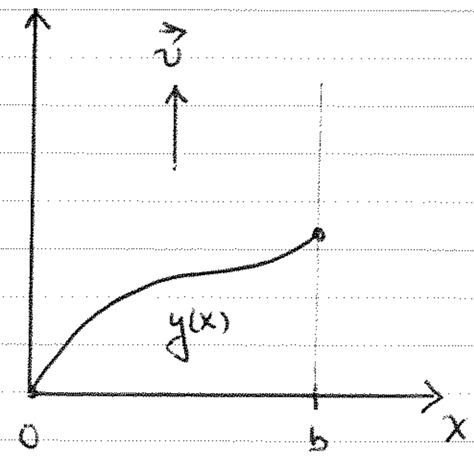
In some instances boundary conditions on admissible functions may not be specified. For example, minimizing

$$J(y) = \int_a^b L(x, y, y')\,dx$$

over the admissible set

$$A = \{ y \in C^2[a,b] : y(a) = 0 \}$$

Here $y(b)$ is not known and part of the problem is to find it. Before we develop theory to find such a "natural boundary condition" we present a real world problem:



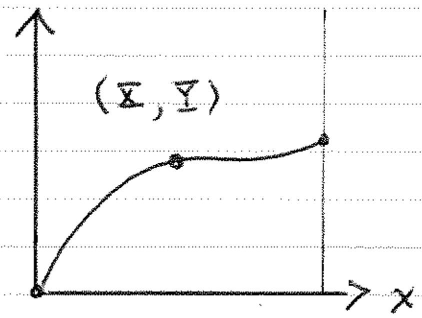
What path should a boat take to minimize the transit time T across a river whose velocity is $\vec{v} = v(x)\hat{j}$?

Here $T = T(y)$ depends on $y(x)$ hence is a functional and the end point $y(b)$ is not known.

Steer problem functional derivation

Assume that if there were no flow the speed of the boat is constant. This would correspond to a constant throttle position say.

Without current



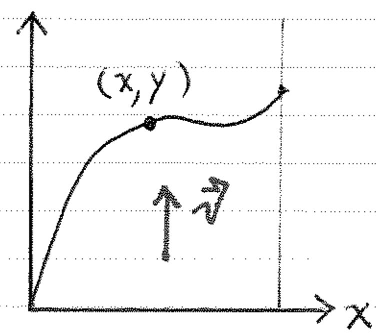
Position at time t

$(\bar{X}(t), \bar{Y}(t))$

Constant speed where we let $\dot{(\ )} = \frac{d}{dt}(\ )$

$$(1) \qquad c = \sqrt{\dot{\bar{X}}^2 + \dot{\bar{Y}}^2}$$

With current



Position at time t

$(x(t), y(t))$

Current Velocity

$\vec{v} = v(x)\,\hat{j}$

Since the current flows in y-direction it only affects the y component of the boat velocity

$$(2) \qquad \dot{x} = \dot{\bar{X}} \qquad \dot{y} = \dot{\bar{Y}} + v$$

Suppose the boat path is $y = y(x)$.
Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Notationally we let $y'(x) = \frac{dy}{dx}$ so that

(3) $\quad \dot{x} = \dot{\underline{X}} \qquad y'(x)\,\dot{x} = \dot{\underline{Y}} + v$

We use (3) in (1) to determine $\dot{x}$

$$c^2 = \dot{\underline{X}}^2 + \dot{\underline{Y}}^2$$

$$c^2 = \dot{x}^2 + (y'\dot{x} - v)^2$$

Expanding this out yields a quadratic equation for $\dot{x}$

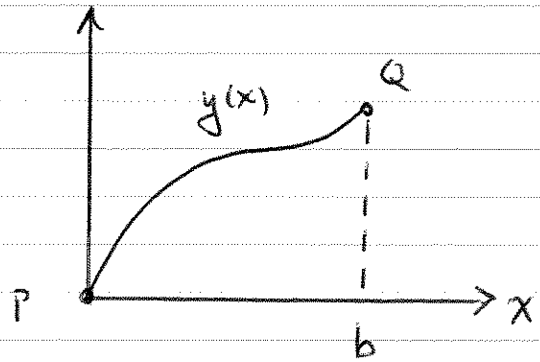
$$(1 + y'^2)\,\dot{x}^2 - 2vy'\dot{x} + (v^2 - c^2) = 0$$

Solving for $\dot{x}$ with quadratic formula

(4) $$\dot{x} = \frac{dx}{dt} = \frac{vy' \pm \sqrt{v^2y'^2 - (1+y'^2)(v^2 - c^2)}}{(1 + y'^2)}$$

We must have $c > v$ else the boat could not cross. With this condition we must take $+$ in (4) so that $\dot{x} > 0$ and the boat crosses.

Transit time T



$$T = \int_P^Q dt = \int_0^b \frac{dt}{dx}\, dx$$

where $\frac{dt}{dx}$ is the reciprocal of $\frac{dx}{dt}$ in eqn (4).

After some algebraic simplification,

$$T(y) = \int_0^b \left(\frac{dx}{dt}\right)^{-1} dx = \int_0^b L(x, y, y')\, dx$$

where

$$L = \frac{\sqrt{c^2(1+y'^2) - v^2} - vy'}{(c^2 - v^2)}$$

and the admissible set is

$$A = \{ y \in C^2[0,b] : y(0) = 0 \}$$

Natural Boundary Conditions

Define

$$J(y) \equiv \int_a^b L(x, y, y')\,dx$$

where the admissible set is

$$\mathcal{A} = C^2[a,b] \qquad \text{(admissible set)}$$

Here no boundary conditions for $y \in \mathcal{A}$ have been stipulated. As a consequence the set of all admissible variations has none either

$$\mathcal{A}^* = C^2[a,b] \qquad \text{(admissible variations)}$$

As before we let $\bar{y} \in \mathcal{A}$ be an extrema and set

$$F(\varepsilon) = J(\bar{y} + \varepsilon h) \qquad h \in \mathcal{A}^*$$

$$F(\varepsilon) = \int_a^b L(x, \bar{y} + \varepsilon h, \bar{y}' + \varepsilon h')\,dx$$

As before

$$F'(\varepsilon) = \int_a^b L_y h + L_{y'} h'\,dx$$

integrate by parts and evaluate at $\varepsilon = 0$ to determine the first variation.

$$F'(0) = \delta J(\bar{y}, h)$$

$$\delta J = L_{y'}(x, \bar{y}(x), \bar{y}'(x))\,h(x)\Big|_{x=a}^{x=b} + \int_a^b \left(L_y - \frac{d}{dx} L_{y'}\right) h\,dx$$

$\uparrow$ boundary terms $\qquad\qquad$ $\uparrow$ integral

If $\bar{y} \in A$ is an extrema of J its first variation must vanish for all $h \in A^*$

This means each of the boundary and integral terms must vanish independently

If we consider only those $h \in A^*$ that vanish at the end points then

$$\delta J = \int_a^b \left(L_y - \frac{d}{dx} L_{y'}\right) h(x)\,dx$$

$\uparrow$ all $h \in A^*$ with $h(a) = h(b) = 0$

This can vanish

$$(1) \qquad L_y - \frac{d}{dx} L_{y'} = 0 \qquad \text{EL-eqn.}$$

, if $\bar{y}$ solves (1)

$$\delta J = L_{y'}(x, \bar{y}(x), \bar{y}'(x))\,h(x)\Big|_{x=a}^{x=b}$$

This must vanish for all $h \in A^*$ including those that do not vanish at $x = a, b$. } KEY POINT

Hence extrema must not only be solns of the EL-eqns but they must also satisfy the B.C.

$$(2) \qquad L_{y'}(a, \bar{y}(a), \bar{y}'(a)) = 0$$

$$(3) \qquad L_{y'}(b, \bar{y}(b), \bar{y}'(b)) = 0$$

Natural Boundary Conds!

Collectively (1)-(3) form a BVP for extrema $\bar{y}(x)$

<u>EXAMPLE</u> (Steering Problem)

$$L(x, y') = (c^2 - v^2)^{-1} (\Delta^{1/2} - v y')$$

where $\Delta = c^2(1+y'^2) - v^2$, $v = v(x)$ and c constant.

Eqn (3) above is natural B.C. for $x = b$.

$$L_{y'} = (c^2 - v^2)^{-1} \left(\frac{c^2 y'}{\Delta^{1/2}} - v^2 \right) = 0$$

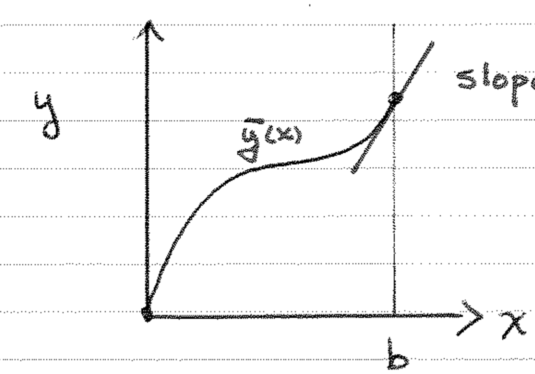
The term in parentheses must vanish $\Rightarrow$

$$c^2 y' = v^2 \Delta^{1/2}$$

$$c^4 y'^2 = v^2 (c^2(1+y'^2) - v^2)$$

Simplify, solve for y' and evaluate at $x = b$

$$\bar{y}'(b) = \frac{v(b)}{c} \qquad \text{Natural B.C.}$$



slope at opposite bank should be that given in Nat. B.C.

<u>EXAMPLE</u> Define $J : \mathcal{A} \to \mathbb{R}$ by

$$J(y) = y(1)^2 + \int_0^1 y'(x)^2\,dx$$

$$\mathcal{A} = \{ y \in C^2[0,1] : y(0) = 1 \}$$

Here the boundary condition at $x = 1$ is "free"
The Lagrangian $L = (y')^2$ and the set of admissible variations is

$$\mathcal{A}^* = \{ h \in C^2[0,1] : h(0) = 0 \}$$

Seek to derive natural B.C. and extrema

$$F(\varepsilon) = J(\bar{y} + \varepsilon h) = (\bar{y}(1) + \varepsilon h(1))^2 + \int_0^1 L(\bar{y}' + \varepsilon h')\,dx$$

Compute $F'(\varepsilon)$

$$F'(\varepsilon) = 2(\bar{y}(1) + \varepsilon h(1))\,h(1) + \int_0^1 L_{y'}(\bar{y}' + \varepsilon h')\,h'\,dx$$

Evaluate at $\varepsilon = 0$ to get first variation.

$$\delta J = 2\bar{y}(1)h(1) + \int_0^1 L_{y'}(\bar{y}')\,h'\,dx$$

Integrate by parts

$$\delta J = (2\bar{y}(1) + L_{y'}(\bar{y}'(1)))\,h(1) - L_{y'}(\bar{y}'(0))\underbrace{\cancel{h(0)}}_{0 \text{ since } h \in \mathcal{A}^*}$$

$$- \int_0^1 \frac{d}{dx} L_{y'}(\bar{y}')\,h(x)\,dx$$

Thus the first variation is

$$\delta J = \underbrace{\left(2\bar{y}(1) + L_{y'}(\bar{y}'(1))\right)}_{=0 \text{ for NBC}} h(1) \quad - \int_0^1 \underbrace{\frac{d}{dx} L_{y'}(\bar{y}')}_{=0 \text{ is EL-eqn}} h(x)\, dx$$

Since $L_{y'} = 2y'$ we conclude extrema must satisfy

$$\begin{array}{llcl} (1) & \frac{d}{dx} L_{y'} & = 0 & \text{EL eqn.} \\ (2) & \bar{y}(0) & = 1 & \text{given B.C.} \\ (3) & \bar{y}(1) + \bar{y}'(1) & = 0 & \text{N.B.C.} \end{array}$$

Explicitly eqn (1) is

$$\frac{d}{dx}(2\bar{y}') = 2\bar{y}'' = 0$$

whose general solution is $\bar{y}(x) = Ax + B$; $A, B \in \mathbb{R}$.

$$\bar{y}'(x) = Ax \quad , \quad \bar{y}(x) = Ax + B$$

Thus, B. Conds are

$$\begin{array}{lcll} \bar{y}(0) & = & B & = 1 \\ \bar{y}(1) + \bar{y}'(1) & = & 2A + B & = 0 \end{array}$$

whose soln is $A = -\frac{1}{2}$, $B = 1$ and the extrema is

$$\bar{y}(x) = -\tfrac{1}{2}x + 1$$