

## Function Expansions - Examples

Let  $f: [a, b] \rightarrow \mathbb{R}$ . For various reasons one may want to represent  $f(x)$  as a series

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad a_n \in \mathbb{R},$$

where  $\phi_n(x)$  are "basis" functions

### EXAMPLE Taylor Series

$$\phi_n(x) = (x - x_0)^n \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

If  $f(x)$  and all its derivatives are continuous near  $x_0$  then Taylor's Thm implies

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

for all  $x$  near  $x_0$ . In particular,

(a) The series converges

(b) The series converges to  $f(x)$  for  $x$  near  $x_0$ .

Some examples

$$e^x = 1 + x + \frac{1}{2!} x^2 + O(x^3) \quad x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + O(x^3) \quad |x| < 1$$

Note  $f(x)$  may be defined for some  $x$  where the series is not, as in  $(1-x)^{-1}$  above!

## EXAMPLE Fourier Series on $[-\pi, \pi]$

Theorem Let  $f(x)$  be piecewise smooth on  $[-\pi, \pi]$  and define

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n \geq 1$$

Then at each  $x \in (-\pi, \pi)$  at which  $f(x)$  is continuous

$$(1) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

This theorem is stated without proof and equality only applies at points of continuity

For  $f(x) = x$  it is easy to show  $a_n = 0 \quad \forall n$  and

$$(2) \quad f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

where here  $\phi_n(x) = \sin(nx)$  are "basis" fns.

For this example  $f(x)$  note  $f(\pi) = \pi$  does NOT equal the series (whose value is zero on account  $\sin(n\pi) = 0$  for all  $n$ ).

## Function Expansion - Issues

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x)$$

A few points / issues

(1) The  $\sim$  as opposed to  $=$  symbol is sometimes to emphasize the fact that  $f(x)$  does not equal the series at some  $x$ .

(2) What is a good choice of basis?

(3) How does one determine the coefficients  $a_n$  once a choice of basis  $\{\phi_n\}$  has been made?

(4) How good an approximation is the truncated series?

$$f(x) \approx \sum_{n=0}^N a_n \phi_n(x)$$

how close are these?

The answers to some involve the study of complete orthogonal bases.

## Square Integrable Functions $L^2[a, b]$

The theoretical framework for series representations of  $f(x)$  is best formulated on the space

$$L^2[a, b] = \left\{ f : \int_a^b f(x)^2 dx < \infty \right\}$$

EXAMPLE Functions  $f(x)$  continuous on the closed bounded interval  $[a, b]$  are in  $L^2[a, b]$

$$f(x) = \sin x \in L^2[0, \pi]$$

EXAMPLE  $f(x) = x^p$ ,  $p \neq \frac{1}{2}$  on  $[0, 1]$

$$\begin{aligned} \int_0^1 f(x)^2 dx &= \frac{1}{2p+1} x^{2p+1} \Big|_{0^+}^1 \\ &= \frac{1}{2p+1} - \frac{1}{2p+1} \lim_{x \rightarrow 0^+} x^{2p+1} \end{aligned}$$

finite only if  $p > -\frac{1}{2}$

Hence  $x^{-1/3} \in L^2[0, 1]$  but  $x^{-1} \notin L^2[0, 1]$

EXAMPLE Unbounded interval  $[a, b] = \mathbb{R} = (-\infty, \infty)$

$$f(x) = e^{-|x|} \in L^2(\mathbb{R}) \text{ since } \int_{\mathbb{R}} e^{-2|x|} dx < \infty$$

$$f(x) = x^2 \notin L^2(\mathbb{R}) \text{ since } \int_{\mathbb{R}} x^2 dx \text{ unbounded}$$

## Inner Products

Let  $\mathbb{X}$  be a linear space. An inner product on  $\mathbb{X}$  is a mapping that associates a real number  $\langle x, y \rangle$  to each ordered pair  $x, y \in \mathbb{X}$  and satisfies

$$(i) \quad \langle x, x \rangle \geq 0 \quad (\text{equality only if } x=0)$$

$$(ii) \quad \langle x, y \rangle = \langle y, x \rangle$$

$$(iii) \quad \langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$(iv) \quad \langle x, \alpha y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}$$

EXAMPLE Dot product on  $\mathbb{X} = \mathbb{R}^3$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

for  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$

EXAMPLE Inner product on  $L^2[a, b]$

$$\langle f, g \rangle \equiv \int_a^b f(x) g(x) dx$$

is an inner product for  $f(x) \in L^2[a, b], g(x) \in L^2[a, b]$

As an example  $f(x) = x, g(x) = x^2$  in  $L^2[0, 1]$

$$\langle f, g \rangle = \int_0^1 x^3 dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4}$$

## Inner product induced norm

Every inner product induces a norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Recall that norms are used to determine distances and lengths:

$$\|x - y\| = \text{distance between } x, y$$

EXAMPLE Norm on  $\mathbb{R}^3$  for  $x = (x_1, x_2, x_3)$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

EXAMPLE Norm on  $L^2[a, b]$ ,  $f(x) \in L^2[a, b]$

$$\|f\| = \sqrt{\langle f, f \rangle} = \left( \int_a^b f(x)^2 dx \right)^{1/2}$$

As an example if  $f(x) = \cos x \in L^2[0, \pi]$

$$\langle f, f \rangle = \int_0^{\pi} \cos^2 x dx = \frac{\pi}{2} \quad \Rightarrow \quad \|f\| = \sqrt{\frac{\pi}{2}}$$

Another example,  $f(x) = x, g(x) = x^2 \in L^2[0, 1]$

$$\|f - g\|^2 = \int_0^1 (x - x^2)^2 dx = \frac{1}{30}$$

so that

$$\|f - g\| = \frac{1}{\sqrt{30}} = \text{"distance between } f, g\text{"}$$

## Orthogonal, Orthonormal

For any inner product space  $x, y \in \mathbb{X}$  are orthogonal if

$$\langle x, y \rangle = 0$$

If both  $x$  and  $y$  are unit length as well they are said to be orthonormal

Every  $x \in \mathbb{X}$  can be normalized by dividing by its length

$$\hat{x} = \frac{x}{\|x\|} \Rightarrow \|\hat{x}\| = 1$$

EXAMPLE  $x = (1, 2, 1)$  and  $y = (-1, 1, -1)$  are orthogonal but not orthonormal in  $\mathbb{R}^3$ , i.e.

$$\langle x, y \rangle = 0 \quad \langle x, x \rangle = 6 \neq 1$$

EXAMPLE  $f(x) = x$   $g(x) = x(\alpha - x)$   $L^2[0, 1]$

$$\langle f, g \rangle = \int_0^1 x^2(\alpha - x) dx$$

$$\langle f, g \rangle = \frac{1}{3}\alpha - \frac{1}{4}$$

Thus  $f$  and  $g$  are  $\perp$  only if  $\alpha = \frac{3}{4}$ .  
Not orthonormal since

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 x^2 dx = \frac{1}{3} \neq 1$$

EXAMPLE Orthonormal set on  $L^2[0, \pi]$

$$f_n(x) = \sin(nx) \quad n = 1, 2, \dots$$

To compute  $\langle f_n, f_m \rangle$  for integers  $n, m$  we use the identity

$$\sin A \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B)$$

For  $n \neq m$  then

$$\begin{aligned} \langle f_n, f_m \rangle &= \frac{1}{2} \int_0^{\pi} \cos((n-m)x) - \cos((n+m)x) dx \\ &= \frac{1}{2(n-m)} \sin((n-m)x) - \frac{1}{2(n+m)} \sin((n+m)x) \Big|_0^{\pi} \\ &= 0 \end{aligned}$$

For  $n = m$

$$\langle f_n, f_n \rangle = \frac{1}{2} \int_0^{\pi} (1 - \cos 2nx) dx = \frac{\pi}{2}$$

so that  $\|f_n\| = \sqrt{\frac{\pi}{2}}$ . Summary

$$\langle f_n, f_m \rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

so  $\{f_n\}$  is an orthogonal set.  $\{\hat{f}_n\}$  is an orthonormal set if

$$\hat{f}_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$



## Pointwise versus $L^2$ (mean square) convergence

Let  $\{f_n\}_{n \geq 1}$  be an orthogonal sequence on  $L^2[a, b]$

$$\langle f_n, f_m \rangle = 0 \quad n \neq m$$

Now consider an  $N$ -term approximation  $S_N(x)$  of  $f(x)$  on  $[a, b]$

$$S_N(x) \equiv \sum_{n=1}^N c_n f(x)$$

QUESTION How well does  $S_N(x)$  approximate  $f(x)$  as  $N$  gets large?

Toward answering this question we define two types of errors

$$E_N(x) \equiv f(x) - S_N(x)$$

$$e_N \equiv \|f - S_N\|^2 = \int_a^b |f(x) - S_N(x)|^2 dx$$

Here

$$E_N(x) = \text{pointwise error at } x \in [a, b]$$

$$e_N(x) = \text{mean square error on } [a, b]$$

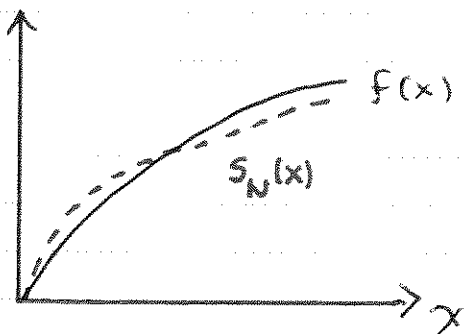
Defn We say  $S_N \rightarrow f$  pointwise at  $x \in [a, b]$  if

$$\lim_{N \rightarrow \infty} (f(x) - S_N(x)) = 0$$

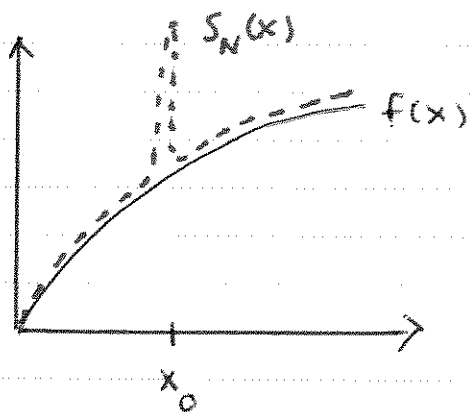
Defn We say  $S_N \rightarrow f$  on  $[a, b]$  in the  $L^2$  or mean square sense if

$$\lim_{N \rightarrow \infty} \|f - S_N\|^2 = 0$$

The difference between these notions of convergence is subtle and best covered in an analysis class. If  $S_N \rightarrow f$  in the  $L^2$  sense it does in the pointwise sense "almost everywhere" which has a very precise meaning (measure theory). To see the difference



Here  $f$  and  $S_N$  are close in both the pointwise and  $L^2$  sense



Here  $f$  and  $S_N$  are close in the  $L^2$  sense but not pointwise at  $x = x_0$

## Generalized Fourier Series

Let  $\{f_n(x)\}_{n \geq 1}$  be an orthogonal sequence.

If there are constants  $c_n$  such that

$$(1) \quad f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

in the  $L^2$  sense then the infinite series is a generalized Fourier series representation of  $f(x)$ .

The constants  $c_n$  are called the Fourier coefficients.

### Finding the Fourier Coefficients from (1)

Multiply (1) by  $f_m(x)$  and integrate

$$\int_a^b f(x) f_m(x) dx = \sum_{n=1}^{\infty} c_n \int_a^b f_n(x) f_m(x) dx$$

$$\langle f, f_m \rangle = \sum_{n \geq 1} c_n \langle f_n, f_m \rangle \quad \begin{array}{l} \nearrow \\ 0 \text{ unless } m=n \end{array}$$

$$\langle f, f_m \rangle = c_m \|f_m\|^2$$

Hence

$$c_m = \frac{\langle f, f_m \rangle}{\|f_m\|^2} \quad \forall m \geq 1$$

## Truncated Series Error $e_N$

Let  $\{f_n\}$  be an orthogonal sequence and  $S_N(x)$  be the  $N$ -term approximation of  $f(x)$ :

$$(1) \quad S_N(x) \equiv \sum_{n=1}^N c_n f_n(x)$$

A natural question to ask is what choice of Fourier coefficients  $c_n$  minimize the  $L^2$  error:

$$e_N = e_N(c_1, \dots, c_N) = \|f(x) - S_N(x)\|^2$$

The error depends on  $c_1, \dots, c_N$ . We use calculus to minimize  $e_N$

$$(2) \quad \frac{\partial e_N}{\partial c_k} = 0 \quad k=1, 2, \dots, N$$

This is  $N$  eqns for  $N$  unknowns  $c_k$ .

$$\begin{aligned} \frac{\partial e_N}{\partial c_k} &= \frac{\partial}{\partial c_k} \int_a^b (f(x) - \sum_{n=1}^N c_n f_n(x))^2 dx \\ &= \int_a^b 2(f(x) - \sum_{n=1}^N c_n f_n(x)) f_k(x) dx \\ &= 2\langle f, f_k \rangle - 2 \sum_{n=1}^N c_n \langle f_n, f_k \rangle \quad \begin{matrix} \nearrow \\ 0 \text{ if } n \neq k \end{matrix} \\ &= 2\langle f, f_k \rangle - 2c_k \|f_k\|^2 \end{aligned}$$

Vanishes only if

$$(2) \quad \boxed{c_k = \frac{\langle f, f_k \rangle}{\|f_k\|^2}}$$

So now we know the choice

$$(3) \quad S_N(x) = \sum_{n=1}^N c_n f_n(x) \quad c_n = \frac{\langle f, f_n \rangle}{\|f_n\|^2}$$

minimizes the  $L^2$  error  $e_N = \|f - S_N\|^2$ .

The next question is how big is this error?

$$\begin{aligned} e_N &= \|f - S_N\|^2 \\ &= \langle f - S_N, f - S_N \rangle \\ &= \langle f, f \rangle - 2 \langle f, S_N \rangle + \langle S_N, S_N \rangle \\ &= \|f\|^2 - 2 \langle f, \sum_{n=1}^N c_n f_n \rangle + \langle \sum_{m=1}^N c_m f_m, \sum_{n=1}^N c_n f_n \rangle \\ &= \|f\|^2 - 2 \sum_{n=1}^N c_n \langle f, f_n \rangle + \sum_{n=1}^N c_n^2 \|f_n\|^2 \\ &= \|f\|^2 - 2 \sum_{n=1}^N c_n^2 \|f_n\|^2 + \sum_{n=1}^N c_n^2 \|f_n\|^2 \end{aligned}$$

So that

$$(4) \quad e_N = \|f\|^2 - \sum_{n=1}^N c_n^2 \|f_n\|^2$$

If additionally  $\{f_n\}$  are normalized, i.e.  $\|f_n\| = 1$

$$(5) \quad e_N = \|f\|^2 - \sum_{n=1}^N c_n^2$$

## Parseval's Equality

If  $\{f_n\}$  is an orthonormal (normalized orthogonal) sequence then we proved

$$(1) \quad e_N = \left\| f(x) - \sum_{n=1}^N c_n f_n(x) \right\|^2 = \|f\|^2 - \sum_{n=1}^N c_n^2$$

for any  $N$  if the Fourier coefficients

$$c_n = \langle f, f_n \rangle, \quad \|f_n\| = 1$$

If  $e_N \rightarrow 0$  as  $N \rightarrow \infty$  (more terms included) then eqn (1) above implies

$$(2) \quad \boxed{\|f\|^2 = \sum_{n=1}^{\infty} c_n^2} \quad \{f_n\} \text{ orthonormal}$$

which is known as Parseval's Equality.

In certain physical settings  $c_n^2$  is the energy of the  $n$ -th "mode" and  $\|f\|^2$  is the total energy.

Equation (2) can be used to derive formulae for certain infinite sums.

EXAMPLE

Previously we showed that

$$f_n(x) = \sin(nx)$$

$$\|f_n\| = \sqrt{\frac{\pi}{2}}$$

form an orthogonal set on  $L^2[0, \pi]$

Let

$$\hat{f}_n = \sqrt{\frac{2}{\pi}} f_n \quad n \geq 1$$

so that  $\{\hat{f}_n\}$  is an orthonormal set.

Assuming that

$$(1) \quad f(x) = x = \sum_{n=1}^{\infty} c_n \hat{f}_n(x)$$

for some  $c_n$ , derive a formula for the sum

$$I = \sum_{n=1}^{\infty} c_n^2$$

First compute the Fourier Coefficients

$$c_n = \langle f, \hat{f}_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^{\pi} x \sin(nx) dx$$

$$c_n = \sqrt{\frac{2}{\pi}} \left( \frac{\overset{0}{\cancel{\sin(nx)}} - nx \cos(nx)}{n^2} \right) \Big|_0^{\pi}$$

$$c_n = \sqrt{\frac{2}{\pi}} \left( -\frac{\pi \cos(n\pi)}{n} \right)$$

$$(2) \quad c_n = \sqrt{2\pi} \frac{(-1)^{n+1}}{n}$$

These are relative to the normalized basis!

## Parseval's Equality

$$\|f\|^2 = \sum_{n=1}^{\infty} c_n^2$$

Given  $f(x) = x$  and the result (2) this becomes

$$\int_0^{\pi} x^2 dx = \sum_{n=1}^{\infty} \left( \sqrt{2\pi} \frac{(-i)^{n+1}}{n} \right)^2$$

$$I = \frac{1}{3} \pi^3 = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

From which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad //$$